Introduction to (dis)continuous Galerkin finite element methods

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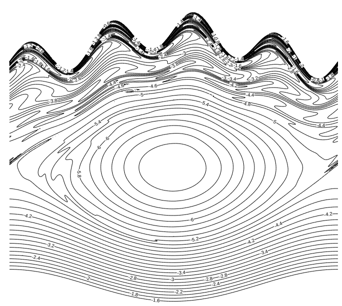

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Introduction

- Finite element methods appear more difficult to understand, but offer more flexibility and superior accuracy in domains with complex boundaries and boundary conditions.
- Furthermore, the mathematical theory of finite element methods is well-developed.
- We will present space and space-time discontinuous Galerkin methods for two basic examples: the linear advection and Burger's equations, as well as their diffusive counterparts.
- For complex applications, see the references.

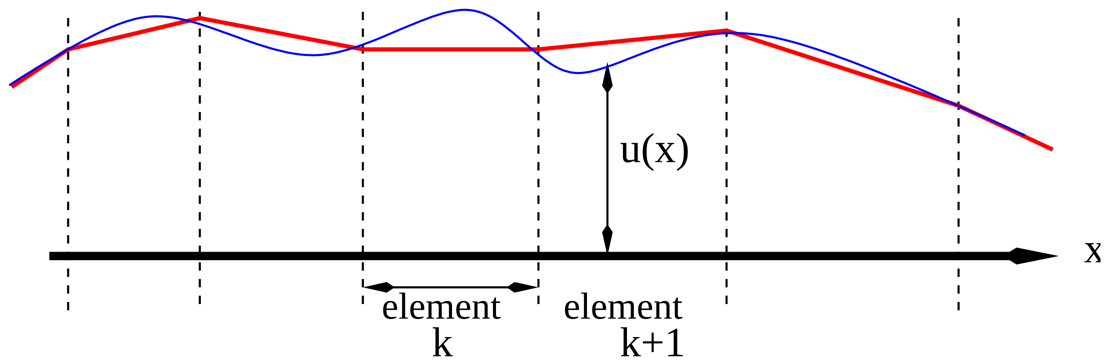
Continuous and discontinuous Galerkin FEM

Consider function $u(x)$ one dimension.

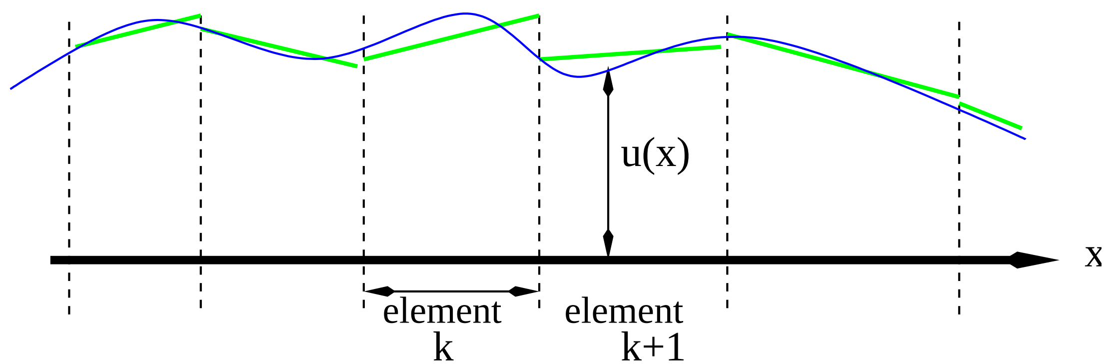
$u(x)$ has piecewise linear approximation $u_h(x)$.

- In continuous FEM, function $u_h(x)$ approximated by piecewise linear function per element. Based one nodal values u_h. Continuous across elements.
- In discontinuous Galerkin FEM, piecewise linear discretization on each element and trace values approaching nodes from the element left or right of node are not continuous.

- Continuous:



- Discontinuous:



FEM algorithm

- Steps in discontinuous Galerkin FE discretization:

  I. *Derive weak formulation:* Each equation multiplied by test function, integrated (by parts) over element, summed over all elements.

  II. *Form discretized weak formulation/algebraic system:* Variables expanded in element using basis functions. Expansion substituted into weak formulation, test function chosen alternately to coincide with a basis function.

  III. *Evaluate of integrals in local coordinate system:* Reference coordinate system used to evaluate the integrals.

  IV. *Solve algebraic system:* Resulting algebraic system solved (iteratively) using forward time stepping methods or linear algebra routines.

Finite elements

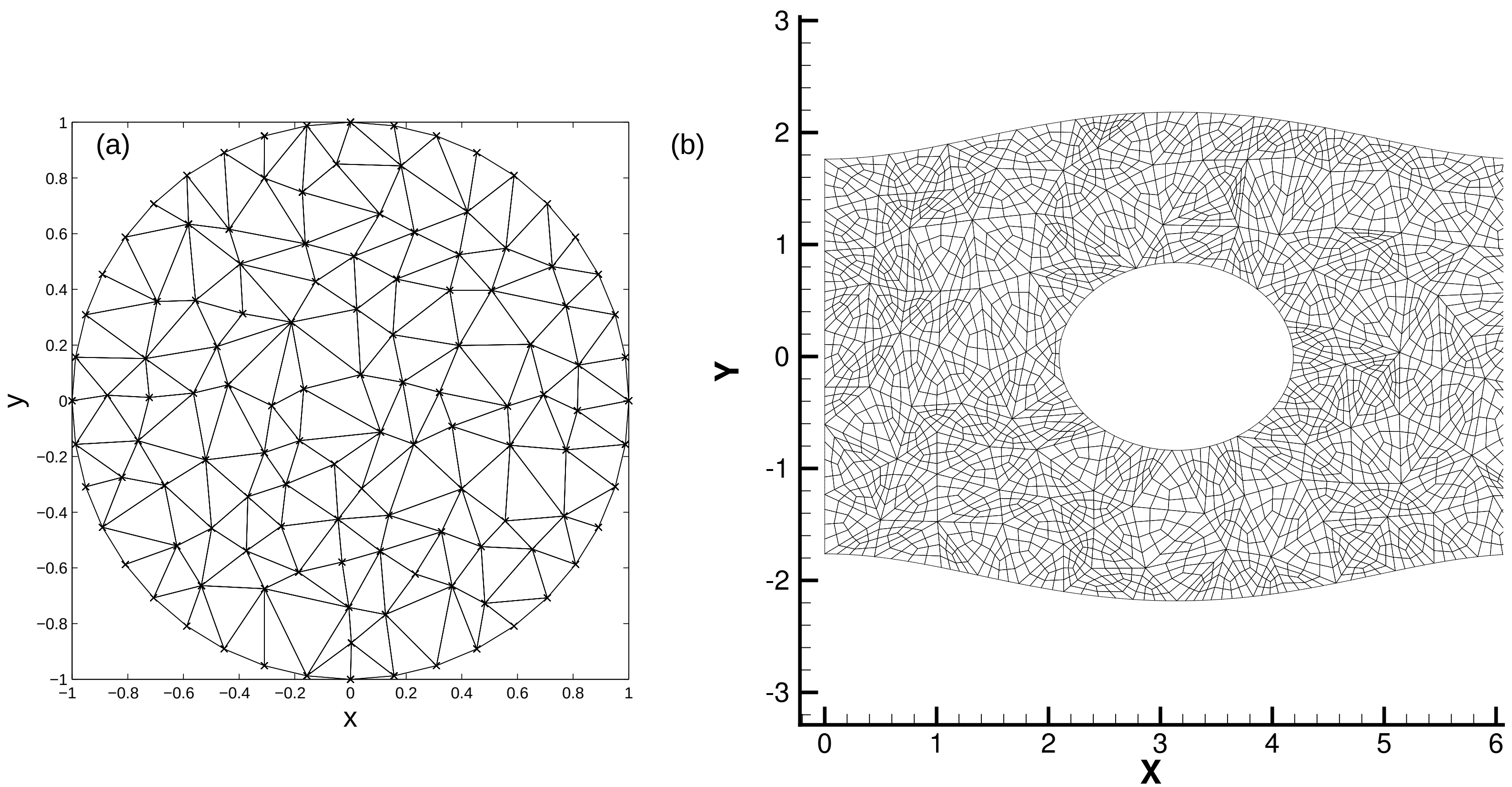
- Algorithm for (pseudo) two-dimensional meshes using Delaunay triangulation.
- Sample programs available by Vijaya Ambati & Erik Bernsen. (Ambati & Bokhove, 2005; Bokhove, Bernsen, & Van der Vegt, 2006).
- Reference coordinates in one dimension.

Pseudo 2D meshes using Delaunay triangulation

Algorithm, triangular base mesh generated as follows.

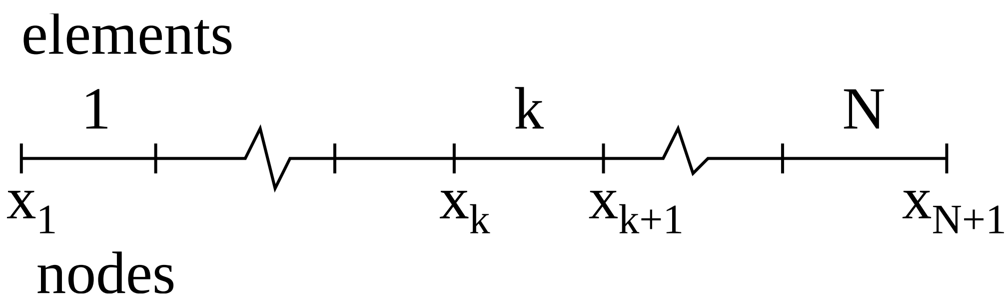
- Boundary points are placed on (circular) domain boundary.
- Specified number of points randomly placed within domain & accepted when more than a critical distance, proportional to the distance between two boundary points, away from all other accepted points.
- Given all points, perform (Matlab) Delaunay triangulation: balanced triangular mesh including assembly information.
- Remove faces that lie outside the domain.
- Enforce definite (counter)clockwise orientation.
- Quadrilateral mesh can be made by placing a point in the middle of each triangle; then divide triangle into three quadrilaterals.

- Resulting quadrilateral mesh can be refined further.
- Orientation and node to element assembly of this division process is subsequently made.
- Courtesy Erik Bernsen



Reference coordinates in one dimension

- A sketch of the one-dimensional finite element mesh with a definition of the node and element numbering. We denote with $N = N_{\text{el}}$ the total number of elements.



- 1D domain $\Omega = x \in [a, b]$ $(b > a; a, b \in \mathbb{R})$ partitioned by points $x_k(t)$, $k = 1, \cdots, N_{\text{el}} + 1$, into N_{el} elements.

- Open elements $K_k = \{x | x \in (x_k, x_{k+1})\}$.

- Tessellation

$$\mathcal{T}_h = \{K_k | \bigcup_{k=1}^{N_{\text{el}}} \bar{K}_k = \bar{\Omega} \text{ and } K_k \cap K_{k'} = \emptyset \text{ if } k \neq k', 1 \leq k, k' \leq N_{\text{el}}\} \tag{1}$$

  with $\bar{K}_k$ the closure of K_k.

- For convenience, also use notation $x_{k,L} := x_k$ and $x_{k,R} := x_{k+1}$.
- We define $|K_k| = x_{k,R} - x_{k,L}$.
- Introduce reference element $\hat{K}$
- Local or reference coordinate $\zeta \in (-1, 1)$ such that

$$x = x(\zeta) = (x_{k+1} + x_k)/2 + |K_k|\,\zeta/2 \quad \text{and} \quad \mathrm{d}x/\mathrm{d}\zeta = |K_k|/2.$$

Space discontinuous Galerkin FEM: examples

- Linear advection and inviscid Burgers' equation, section 4.1.1
- Advection-diffusion and viscous Burgers' equation, section 4.1.2
- Numerical exercise 4.3.1 on linear advection and Burger's equation

Linear advection and inviscid Burgers' equation

- Consider in particular linear advection equation

$$\partial_t u + a\,\partial_x u = 0 \tag{2}$$

  for $a \in \mathbb{R}$, and

- inviscid Burgers' equation

$$\partial_t u + u\,\partial_x u = 0 \tag{3}$$

- Initial condition $u_0 = u(x, 0)$.
- Boundary conditions for linear advection equation either (i) periodic, or (ii) specified at the inflow boundary:

$$\begin{aligned} u(0,t) =& u_{left}(t) \quad \text{if} \quad a > 0 \qquad \text{or} \\ u(L,t) =& u_{right}(t) \quad \text{if} \quad a \le 0. \end{aligned} \tag{4}$$

- Boundary conditions for the Burgers' equation either (i) periodic, (ii) specified at inflow boundaries:

$$\begin{aligned} u(0,t) =& u_{left}(t) \quad \text{if} \quad u(0,t) > 0 \qquad \text{or} \\ u(L,t) =& u_{right}(t) \quad \text{if} \quad u(L,t) \leq 0, \end{aligned} \tag{5}$$

or (iii) solid walls with $u(0,t) = u(L,t) = 0$.

- Concisely combine (2) and (3) as follows

$$\partial_t u + \partial_x f = 0 \tag{6}$$

- flux $f = f(u)$ in general,
- $f(u) = a\,u$ for the linear advection equation,
- and $f(u) = u^2/2$ for Burgers' equation, in particular, with initial condition $u_0(x) = u(x,0)$.

Weak formulation

- Multiply (6) by arbitrary test function $w = w(x)$ (smooth within each element)
- integrate by parts over each individual and isolated element
- add the contribution from all elements to obtain
- weak formulation:

$$\sum_{k=1}^{N_{\mathrm{el}}} \left\{ \int_{K_k} w \, \frac{du}{dt} \, \mathrm{d}x + [f(x_{k+1}^-) \, w(x_{k+1}^-) - f(x_k^+) \, w(x_k^+)] - \int_{K_k} f(u) \, \partial_x w \, \mathrm{d}x \right\} = 0, \tag{7}$$

- $w(x_{k+1}^-) = \lim_{x \uparrow x_{k+1}} w(x,t)$ and $w(x_k^+) = \lim_{x \downarrow x_k} w(x,t)$.
- Let $[u] = u_+ - u_-$ and $\bar{u} = (u_+ + u_-)/2$ denote jump and mean in the quantity u.

- $u_- = \lim_{x \uparrow x_k} u(x)$ and $u_+ = \lim_{x \downarrow x_k} u(x)$.
- $u_- := u(x_{k+1}^-) \neq u(x_{k+1}^+) =: u_+$ and consequently
- flux $f(x_{k+1}^-) \neq f(x_{k+1}^+)$ in general.
- Rewrite WF (7) sum of interior integrals over nodes

$$\sum_{k=1}^{N_{\text{el}}} \left\{ \int_{K_k} w \, \frac{du}{dt} \, \mathrm{d}x - \int_{K_k} f(u) \, \partial_x w \, \mathrm{d}x \right\} - f(x_1^+) \, w(x_1^+)$$
$$+ f(x_{N_{\text{el}}+1}^-) \, w(x_{N_{\text{el}}+1}^-)$$
$$+ \sum_{k=2}^{N_{\text{el}}} \big(f(x_k^-) \, w(x_k^-) - f(x_k^+) \, w(x_k^+) \big) = 0. \tag{8}$$

- Rewrite flux term at interior nodes:

$$
\begin{aligned}
f(x_k^-)\, w(x_k^-) - f(x_k^+)\, w(x_k^+) &= - \big((\gamma_1\, f_+ + \gamma_2\, f_-)\, [w] \\
&\quad + [f]\, (\gamma_2\, w_+ + \gamma_1\, w_-)\big) \\
&\stackrel{[f]=0}{=} - (\gamma_1\, f_+ \gamma_2\, f_-)\, [w] \qquad (9)
\end{aligned}
$$

  if we enforce continuity, i.e. $[f] = 0$, at a node; also $\gamma_1 + \gamma_2 = 1$ and $\gamma_{1,2} \geq 0$.

- Replace $f(x_k^-)$ and $f(x_k^+)$ both by the same numerical flux $\tilde{f} = \tilde{f}(u_-, u_+) = \gamma_1\, f_+ + \gamma_2\, f_-$.

- In general, numerical flux chosen as function of u_+ and u_-:

$$
\tilde{f} = \tilde{f}(u_-, u_+) = \tilde{f}\big(u_-(x_k), u_+(x_k)\big).
$$

Discretized weak formulation

- Introduce approximations u_h, w_h to $u = u(x,t)$ and test functions $w = w(x)$.
- u_h and w_h belong to broken space

$$V_h = \Big\{ v \Big| v|_{K_k} \in P^{d_P}(K_k), k = 1, \ldots, N_{\text{el}} \Big\}, \tag{10}$$

  in which $P^{d_P}(K_k)$ space of polynomials in K_{k} of degree d_P.

- Note that u_h is continuous in element but generally discontinuous across element boundaries.

- Discretized weak formulation

$$\sum_{k=1}^{N_{\mathrm{el}}}\left\{\int_{K_k} w_h\,\frac{\mathrm{d}u_h}{\mathrm{d}t}\,\mathrm{d}x-\int_{K_k} f(u_h)\,\partial_x w_h\,\mathrm{d}x\right\}$$
$$-\tilde{f}\left(u_{left},u_h(x_1^+)\right)\,w_h(x_1^+)$$
$$+\tilde{f}\left(u_h(x_{N_{\mathrm{el}}+1}^+),u_{right}\right)\,w_h(x_{N_{\mathrm{el}}+1}^-)$$
$$+\sum_{k=1}^{N_{\mathrm{el}}}\tilde{f}\left(u_h(x_k^-),u_h(x_k^+)\right)\,\left(w_h(x_k^-)-\,w_h(x_k^+)\right)=0 \qquad (11)$$

- Here, we approximate u_h, w_h on K_k by a mean and a slope

$$\begin{aligned} u_h(x,t) =& \bar{U}_k + \hat{U}_k\,\psi_{1,k}(x) \quad \text{and} \\ w_h(x) =& \bar{W}_k + \hat{W}_k\,\psi_{1,k}(x) \end{aligned} \tag{12}$$

- $\bar{U}_k = \bar{U}(K_k, t)$ the mean
- $\hat{U}_k = \hat{U}_1(K_k, t)$ the slope. We note that $\psi_{1,k} = \zeta$.

- Use arbitrariness of $\bar{W}_k$ and $\hat{W}_k$ to derive ODE's for means and slopes per element

$$|K_k| \frac{\mathrm{d}\bar{U}_k}{\mathrm{d}t} + \tilde{f}(x_{k+1}) - \tilde{f}(x_k) = 0 \tag{13a}$$

$$\frac{|K_k|}{3} \frac{\mathrm{d}\hat{U}_k}{\mathrm{d}t} + [\tilde{f}(x_{k+1}) + \tilde{f}(x_k)] - \int_{-1}^{1} f(U_h) \, \mathrm{d}\zeta = 0. \tag{13b}$$

- Integrals approximated Gauss quadrature rule

$$\int_{-1}^{1} f(\zeta) \, \mathrm{d}\zeta \approx f(-c_m) + f(c_m) \tag{14}$$

$c_m = 1/\sqrt{3}$ for some function $f = f(\zeta)$.

Numerical flux

The numerical flux, $\tilde{f}(u_-, u_+)$, is chosen to

- (i) be consistent such that $\tilde{f}(u,u) = f(u)$,
- (ii) be conservative, as used in derivation in (9), and
- (iii) reduce to an E-flux, that is,

$$\int_{u_-}^{u_+} f(s) - \tilde{f}(u_-, u_+)\,\mathrm{d}s \geq 0.$$

- This last property of the E-flux guarantees L^2-stability as we will discuss later.
- Note that numerical flux is only way of communication between elements, and that flux determined by values of u_h immediately left and right of each node.

- Given u_- and u_+ immediately left and right of node x_k, we wish to obtain a numerical flux.
- Extend the values $u_\pm$ into the left and right elements.
- Calculate exact solution local Riemann problem around node x_k: find solution $u = u_e(x, t)$ for $t > t_0$ of

$$\partial_t u + \partial_x f(u) = 0 \tag{15}$$

$$u(x, t_0) = \begin{cases} u_- & x < x_k \\ u_+ & x \geq x_k \end{cases} . \tag{16}$$

- Numerical flux is then defined as $\tilde{f} = f(u_e(x_k, t))$.

Linear advection equation

- Linear advection equation characteristics $\mathrm{d}x/\mathrm{d}t = a$.
- $\mathrm{d}u/\mathrm{d}t = 0$ on $x = x_0 + a\,t$; x_0 integration constant.
- Solution Riemann problem (15) is upwind solution:

$$u_e(x,t') = \begin{cases} u_- & x < x_k + a\,t' \\ u_+ & x \geq x_k + a\,t' \end{cases} \tag{17}$$

  for $t' = t - t_0$.

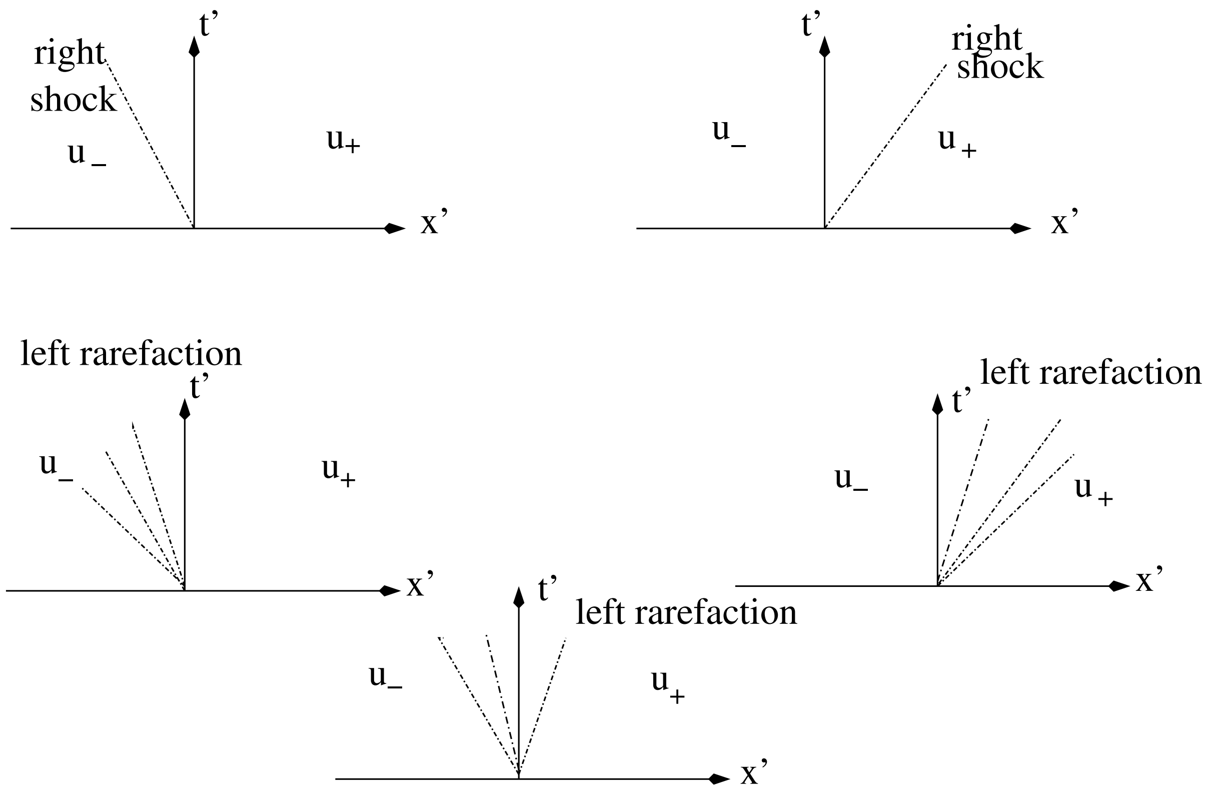
- Numerical flux:

$$\tilde{f}_{linear\ advection}(x_k) = \begin{cases} a\,u_- & a > 0 \\ a\,u_+ & a \leq 0 \end{cases} . \tag{18}$$

Burger's equation

- Characteristics of Burgers' equation are $\mathrm{d}x/\mathrm{d}t = u$.
- Hence, $\mathrm{d}u/\mathrm{d}t = 0$ on $x = x_0 + u_0\,t$.
- Solution is implicit $u(x,t) = u_0(x - u(x,t))$.
- For constant initial data in Riemann problem, characteristics readily solved.
- Shock wave when characteristics converge for $u_- > u+$.
- Rarefaction wave when characteristics diverge for $u_- \leq u_+$.

- Sketch:



- Discontinuity of the Burgers’ shock at $x_b(t)$.
- Has speed $s = \mathrm{d}x_b(t)/\mathrm{d}t$ (the Rankine Hugoniot relation):

$$0 = \lim_{\epsilon \to 0} \int_{x_b(t)-\epsilon}^{x_b(t)+\epsilon} \partial_t u + \partial_x (u^2/2) \, \mathrm{d}x = -s\,[u] + [u^2/2] \qquad (19)$$

with $\bar{u} = (U_l + U_r)/2$ and $[u] = U_r - U_l$, and U_l and U_r the values immediately left and right of the shock.

- Hence, $s = (U_l + U_r)/2$.
- In Riemann problem shock speed $s = (u_- + u_+)/2$ and position $x' = s\,t'$.
- Flux evaluated at $x' = x - x_k$; flux is either $\tilde{f} = u_-^2/2$ when $s > 0$, or $\tilde{f} = u_+^2/2$ when $s \leq 0$ for shock wave case.
- Rarefaction wave in Riemann problem has characteristics $\mathrm{d}x'/\mathrm{d}t' = u$ on which u is constant
- Tail and head of rarefaction wave lie at $x' = u_-\,t'$ and $x' = u_+\,t'$.

- Hence solution is

$$u(x', t') = \begin{cases} u_- & x' < u_- t' \\ x'/t' & u_- t' < x' < u_+ t' \\ u_+ & x' > u_+ t' \end{cases} . \qquad (20)$$

- Hence, numerical flux Burgers's equation:

$$\tilde{f}_{burgers}(x_k) = \begin{cases} u_-^2/2 & s > 0 \wedge u_- > u_+ \\ u_+^2/2 & s \leq 0 \wedge u_- > u_+ \\ u_-^2/2 & u_- > 0 \wedge u_- \leq u_+ \\ 0 & u_- < 0 \wedge u_+ > 0 \wedge u_- \leq u_+ \\ u_+^2/2 & u_+ < 0 \wedge u_- \leq u_+ \end{cases} . \qquad (21)$$

Initial and boundary conditions

- Projection of initial condition onto coefficients:

$$\bar{U}_k(0) = \frac{1}{2} \int_{-1}^{1} u_0\left(x(\zeta)\right)\, \mathrm{d}\zeta \tag{22}$$

$$\hat{U}_k(0) = \frac{3}{2} \int_{-1}^{1} u_0\left(x(\zeta)\right)\, \zeta\, \mathrm{d}\zeta. \tag{23}$$

Time discretization and time step

- Write (13) as system of ODE's

$$\frac{\mathrm{d}\mathbf{U}}{\mathrm{d}t} = \mathbf{G}(\mathbf{U}, t) \tag{24}$$

  with $\mathbf{U} = (\bar{U}, \hat{U})^T$ state vector of unknown coefficients & rest in $\mathbf{G}$ on RHS

- Second-order Runge-Kutta scheme Osher and Shu (1989):

$$\begin{aligned} \mathbf{U}^{(1)} &= \mathbf{U}^n + \Delta t\, \mathbf{G}(\mathbf{U}^n, t^n) \\ \mathbf{U}^{n+1} &= \Big[\mathbf{U}^n + \mathbf{U}^{(1)} + \Delta t\, \mathbf{G}(\mathbf{U}^{(1)}, t^n + \Delta t)\Big]/2 \quad \text{or} \end{aligned} \tag{25}$$

- Third-order Runge-Kutta scheme Osher and Shu (1989):

$$\begin{aligned}\mathbf{U}^{(1)} =& \mathbf{U}^n + \Delta t\, \mathbf{G}(\mathbf{U}^n, t^n) \\ \mathbf{U}^{(2)} =& \Big[3\, \mathbf{U}^n + \mathbf{U}^{(1)} + \Delta t\, \mathbf{G}(\mathbf{U}^{(1)}, t^n + \Delta t)\Big]/4 \\ \mathbf{U}^{n+1} =& \Big[\mathbf{U}^n + 2\, \mathbf{U}^{(2)} + 2\, \Delta t\, \mathbf{G}(\mathbf{U}^{(2)}, t^2 + \Delta t/2)\Big]/3\end{aligned} \tag{26}$$

Time step

- Numerical flux defined in (18) and (21), evaluated at $t' = 0$, so latest (intermediate) value of $u_h(x,t)$ used to define $u_\pm$.
- Time step estimate based on characteristics:

$$\begin{aligned}\Delta t =& CFL \min_k(|K_k|)/|a| \qquad \text{or} \\ \Delta t =& CFL \min_k(|K_k|)/\max_k(|\bar{U}_k^n + \hat{U}_k^n|, |\bar{U}_k^n - \hat{U}_k^n|)\end{aligned} \tag{27}$$

  with CFL the Courant-Friedrichs-Lewy number; $CFL \leq 1$. Time step can vary over time.

- Via linear stability analysis of linear advection equation more detailed time step calculated or estimated.

Pseudo-code

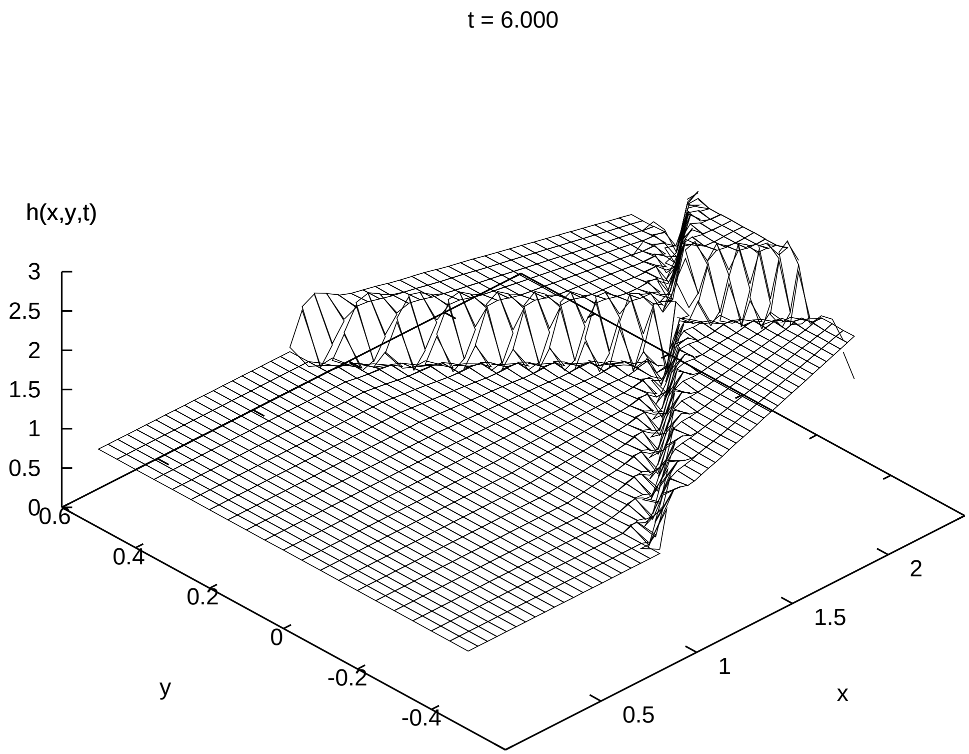
Program outline for the discontinous Galerkin FEM of 1D hyperbolic equation (6) is:

read input file/make input: get $N_{\rm el}, CFL, T_{end}, \ldots$
read mesh file/make mesh: make $|K_k|, x_k, \bar{U}_k, \hat{U}_k, \ldots, \tilde{f}_k, RHS_k$
set initial condition (22)
while $(time < T_{end})$ time loop (to solve (13a),(13b)),
. determine time step
. do intermediate time steps, e.g. RK3 (26)
. calculate flux $\tilde{f}_k$ at nodes x_k, see, e.g., (18) & (21)
. calculate element integrals in (13b), put in RHS_k
. solution update
. do measurements

$End.$ (28)

Animations

- Research on two-dimensional vortical flows with separatrices: Bernsen, Bokhove & Vanneste (2004).
- Shallow water flow through a contraction:



Numerical exercise 4.3.1

Linear advection and Burger's equation

- Goal is numerical implementation and verification of the space discontinuous FEM for the linear advection and inviscid Burgers' equations
- Well-known strategy: test code first for mean values only (13a).

Advection-diffusion and viscous Burgers' equation

- Linear advection diffusion equation

$$\partial_t u + a\,\partial_x u = \kappa\,\partial^2_{xx} u \tag{29}$$

for $a, \kappa \in \mathbb{R}$ and $\kappa > 0$.

- Viscous Burgers' equation

$$\partial_t u + u\,\partial_x u = \kappa\,\partial^2_{xx} u. \tag{30}$$

- Domain $\Omega = [0, L]$ with initial condition $u_0 = u(x, 0)$.
- Combine (29) and (30) into system

$$\partial_t u + \partial_x F_u = 0 \qquad \text{and} \qquad \kappa\,q + \partial_x F_q = 0 \tag{31}$$

with fluxes $F = (F_u, F_q)^T$:

$$F_u = F_u(u, q) = f(u) + \kappa\,q \qquad \text{and} \qquad F_q = F_q(u) = \kappa\,u. \tag{32}$$

- flux F_u: convective part $f(u)$ and diffusive part $\kappa\, q$.
- Advection diffusion equation: $f(u) = a\, u$.
- Burger's equation: $f(u) = u^2/2$.
- We consider periodic boundary conditions.

Weak formulation WF

- Approximations, $w_h = (u_h, q_h)$, to the state vector, (u, q).
- Multiply (31) by $v = \big(v_u(x), v_q(x)\big)$,. . . ,$\rightarrow$ WF:

$$\sum_{k=1}^{N_{\text{el}}} \Big\{ \int_{K_k} v_u \frac{\mathrm{d}u_h}{\mathrm{d}t} \,\mathrm{d}x + \big(\tilde{F}_u(x_{k+1})\, v_u(x_{k+1}^-) - \tilde{F}_u(x_k)\, v_u(x_k^+)\big)$$
$$- \int_{K_k} F_u \,\partial_x v_u \,\mathrm{d}x \Big\} = 0$$
$$\sum_{k=1}^{N_{\text{el}}} \Big\{ \int_{K_k} \kappa\, v_q\, q_h \,\mathrm{d}x + \big(\tilde{F}_q(x_{k+1})\, v_q(x_{k+1}^-) - \tilde{F}_q(x_k)\, v_q(x_k^+)\big)$$
$$- \int_{K_k} F_q \,\partial_x v_q \,\mathrm{d}x \Big\} = 0,$$

- $v_{u,q}(x_{k+1}^-) = \lim_{x\uparrow x_{k+1}} v_{u,q}(x,t)$ & $v_{u,q}(x_k^+) = \lim_{x \downarrow x_k} v_{u,q}(x,t)$.
- Numerical $\tilde{F}_u(x_k) = \tilde{F}_u(w_k^-, w_k^+)$ and $\tilde{F}_q(x_k) = \tilde{F}_q(u_k^-, u_k^+)$.

Weak formulation

- We approximate u_h and q_h by a mean and slope

$$u_h(x,t) = \bar{U}_k + \hat{U}_k\,\zeta \quad \text{and} \quad q_h(x) = \bar{Q}_k + \hat{Q}_k\,\zeta \tag{33}$$

  with $\bar{Q}_k = \bar{Q}(K_k, t)$ the mean and $\hat{Q}_k = \hat{Q}_k(K_k, t)$ the slope, and likewise for v_u, v_q.

- Arbitrariness of the test functions, . . . , $\rightarrow$ system of coupled algebraic and ordinary equations:

$$
\begin{aligned}
|K_k| \frac{\mathrm{d}\bar{U}_k}{\mathrm{d}t} + \tilde{F}_u(x_{k+1}) - \tilde{F}(x_k) &= 0 \\
\frac{|K_k|}{3} \frac{\mathrm{d}\hat{U}_k}{\mathrm{d}t} + \left(\tilde{F}_u(x_{k+1}) + \tilde{F}_u(x_k)\right) - \int_{-1}^{1} F_u(u_h, q_h)\, \mathrm{d}\zeta &= 0 \\
\kappa\, |K_k|\, \bar{Q}_k + \tilde{F}_q(x_{k+1}) - \tilde{F}_q(x_k) &= 0 \\
\kappa\, \frac{|K_k|}{3}\, \hat{Q}_k + \left(\tilde{F}_q(x_{k+1}) + \tilde{F}_q(x_k)\right) - \int_{-1}^{1} F_q(u_h, q_h)\, \mathrm{d}\zeta &= 0.
\end{aligned}
\tag{34}
$$

- Integrals $\rightarrow$ third-order Gauss quadrature rule (14).

Local discontinuous Galerkin method: numerical flux

The numerical flux, $\tilde{F}_u(w_-, w_+)$, is chosen to

- (i) be consistent such that $\tilde{F}(w, w) = F(w)$,
- (ii) be conservative,
- (iii) ensure a local determination of q_h in terms of u_h,
- (iv) reduce to an E-flux in the conservative limit when $\kappa = 0$, that is,

  $$\int_{u_-}^{u_+} F_u(s, q; \kappa = 0) - \tilde{F}_u(w_-, w_+; \kappa = 0)\, \mathrm{d}s \geq 0$$

  with $F_u(u, q; \kappa = 0) = f(u)$, and
- (v) be L^2-stable, as will be shown.

Flux and L^2 stability

- The numerical flux chosen is

$$\tilde{F}_u(w_-, w_+) = \tilde{f}(u_-, u_+) + \kappa\, q_+ \quad \text{and} \quad \tilde{F}_q = F_q(u_-) = \kappa\, u_-, \tag{35}$$

- Convective flux $\tilde{f}(u_-, u_+)$ valid in the inviscid limit $\kappa = 0$.
- Diffusive flux is alternating, $\kappa\, q_+$ in $\tilde{F}_u$ versus u_- in $\tilde{F}_q$.
- L^2 stability. L^2-stability for the discretized equations follows in an analogy of the L^2-stability for the continuous case, and motivates choice of numerical flux.
- Continuous case: multiply (31) by u and q, sum, and integrate

over space and time to obtain:

$$\frac{1}{2}\int_0^L u(T)^2 - u_0^2\,\mathrm{d}x + \int_0^T\int_0^L \kappa\, q^2\,\mathrm{d}x\,\mathrm{d}t$$
$$-\int_0^T\int_0^L (F_u\,\partial_x u + F_q\,\partial_x q)\,\mathrm{d}x\,\mathrm{d}t +$$
$$\int_0^T (u\,F_u + q\,F_q)_{x=L} - (u\,F_u + q\,F_q)_{x=0}\ \mathrm{d}t = 0 \iff$$
$$\frac{1}{2}\int_0^L u(T)^2 - u_0^2\,\mathrm{d}x + \int_0^T\int_0^L \kappa\, q^2\,\mathrm{d}x\,\mathrm{d}t + \int_0^T \big(u\,F_u - \phi(u)\big)|_{x=L}$$
$$- \big(u\,F_u - \phi(u)\big)|_{x=0}\,\mathrm{d}t, \tag{36}$$

- since

$$F_u\,\partial_x u + F_q\,\partial_x q = f(u)\,\partial_x u + \kappa\, q\,\partial_x u + \kappa\, u\,\partial_x q = \partial_x(\phi(u) + q\,F_q).$$

- with $\phi(u) = \int_0^u f(s)\,\mathrm{d}s$.
- Hence, *grosso modo boundary conditions*

$$\frac{1}{2}\int_0^L u(T)^2 - u_0^2\,\mathrm{d}x + \int_0^T \int_0^L \kappa\, q^2\,\mathrm{d}x\,\mathrm{d}t = 0. \qquad (37)$$

Outline discrete L^2 stability

Discrete version of L^2-stability for the numerical spatial discretization proceeds along the same lines (*cf.*, Cockburn and Shu, 1998):

- Add the weak formulation (33) of both equations and integrate in time.
- As in continuous case, substitute $v_u = u_h$ and $v_q = q_h$ and reorder.
- Goal is to prove that, *grosso modo boundary conditions*,

$$\frac{1}{2}\int_0^L (u_h(T)^2 - u_h(x,0)^2)\,\mathrm{d}x + \int_0^T\int_0^L \kappa\, q_h^2\,\mathrm{d}x\,\mathrm{d}t \le 0. \quad (38)$$

Time discretization

- RK2 or RK3 or Crank-Nicolson.
- Abstract system:

$$\frac{\mathrm{d}\mathbf{U}}{\mathrm{d}t} = \mathbf{G}_U(\mathbf{U}, \mathbf{q}) \qquad \text{and} \qquad \mathbf{q} = \mathbf{G}_q(\mathbf{U}) \tag{39}$$

with $\mathbf{U} = (\bar{U}, \hat{U})^T$ state vector unknown coefficients and $\mathbf{q} = (\bar{Q}, \hat{Q})^T$, and $\mathbf{G}_{U,q}$ rest terms on RHS.

- RK2 or Crank-Nicolson; or RK3:

$$\mathbf{q}^n = \mathbf{G}_q(\mathbf{U}^n), \quad \mathbf{U}^{(1)} = \mathbf{U}^n + \Delta t\, \mathbf{G}_U(\mathbf{U}^n, \mathbf{q}^n)$$

$$\mathbf{q}^{(1)} = \mathbf{G}_q(\mathbf{U}^{(1)}), \quad \mathbf{U}^{(2)} = \frac{\Big(3\,\mathbf{U}^n + \mathbf{U}^{(1)} + \Delta t\, \mathbf{G}_U(\mathbf{U}^{(1)}, \mathbf{q}^{(1)})\Big)}{4}$$

$$\mathbf{q}^{(2)} = \mathbf{G}_q(\mathbf{U}^{(2)}), \quad \mathbf{U}^{n+1} = \frac{\Big(\mathbf{U}^n + 2\,\mathbf{U}^{(2)} + 2\,\Delta t\, \mathbf{G}_U(\mathbf{U}^{(2)}, \mathbf{q}^{(2)})\Big)}{3}. \tag{40}$$

- Solve for **U** and **q** in an explicit manner because new stage of **q** found before commencing the time update.

- Time step: do linear stability analysis or use rough estimate based on FD analysis.

Other diffusive systems and fluxes

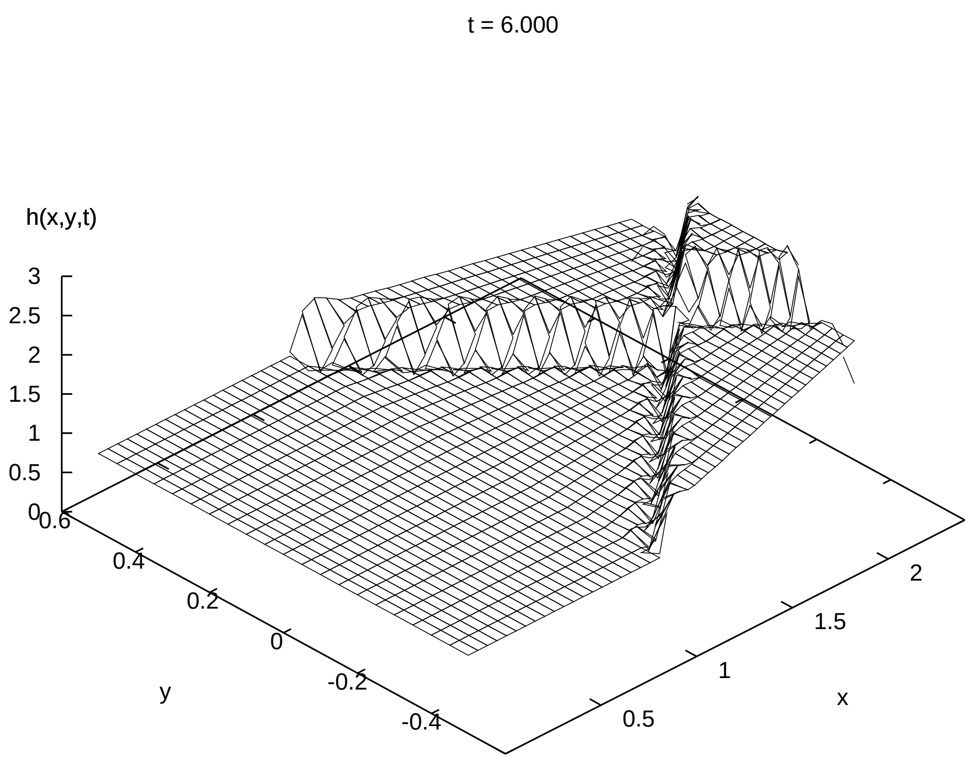
- Nonlinear advection diffusion equation, local discontinuous Galerkin FEM: see original article of Cockburn and Shu (1998). Example: Bokhove, Woods, and De Boer (2005).

- Brezzi technique: Arnold, Brezzi, Cockburn, & Marini (*Siam J. Numer. Anal.*, 2002). No additional variables required.

Animations

- Research on two-dimensional vortical flows with separatrices: Bernsen, Bokhove & Vanneste (2004).

- Shallow water flow through a contraction:



References

Applications of discontinuous Galerkin methods in (more) complex 1D, 2D, & 3D applications developed in our group are:

- vortical geophysical flows

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- Chemical etching
  Sudirham, J.J., Van der Vegt, J.J.W. and Van Damme, R.M.J., (2004) 'Space-time discontinuous Galerkin method for advection-diffusion problems on time-dependent domains'. Submitted to *Appl. Numer. Math.*