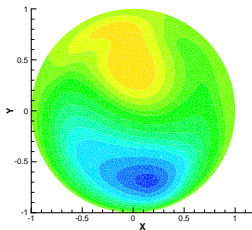
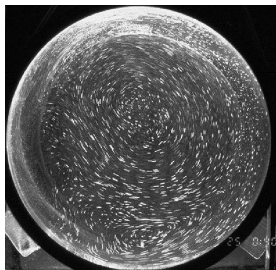


# Simulations of hybrid Rossby-shelf modes in a laboratory ocean



Vijaya Ambati, Onno Bokhove 2009: J. Phys. Ocean. 39 Pedlosky collection

Van der Vegt, Iszak, B. 2007: Error analysis. SIAM J. Num. Anal. 45

Bernsen, B., Van der Vegt 2006: J. Comp. Phys. 211

B., Johnson 1999: J. Phys. Ocean. 29

**Numerical Analysis and Computational Mechanics Group**

**Department of Applied Mathematics, University of Twente**

**Numdiff-12, September 2009**

# Motivation

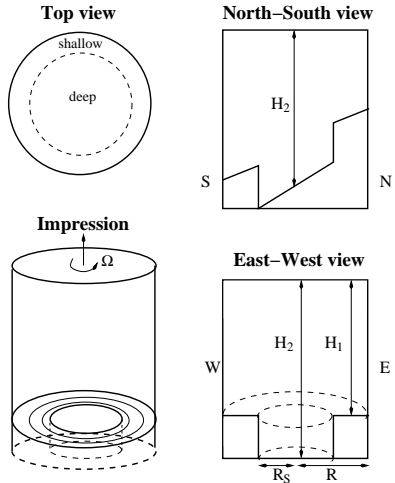
- Explore difference between linear theory and laboratory experiments with **simulations** of nonlinear model.
- Generally: consider wave coupling deep ocean and shallow seas: **relevance of wave damping along coasts to climate**.
- Nonlinear model: (dis)continuous Galerkin finite elements (**DCGFEM**) for **generalized vorticity-streamfunction** or hyperbolic-elliptic system.
- Meanwhile **conserving vorticity and energy**, thus maintaining stability for enstrophy.
- DCGFEM allows unstructured grids in **complex-shaped domains**.

# Outline

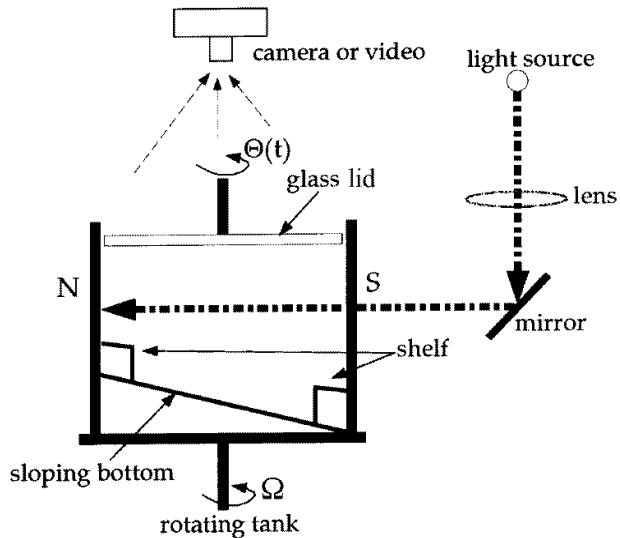
- Rotating tank experiments
- 2D Model
- Continuous-Discontinuous Galerkin Finite Element Method
- Validation
- Conclusion

# Rotating tank experiments

- Abrupt coastal shelf topography
- & deep interior ocean and shallow shelf seas.

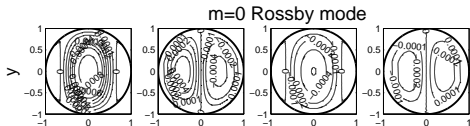
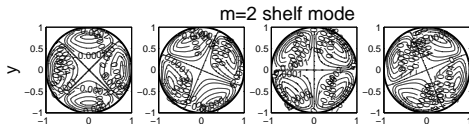
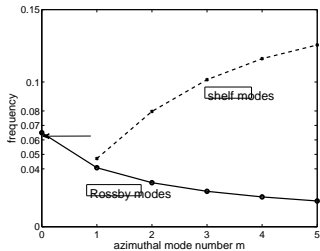


# Rotating tank experiments



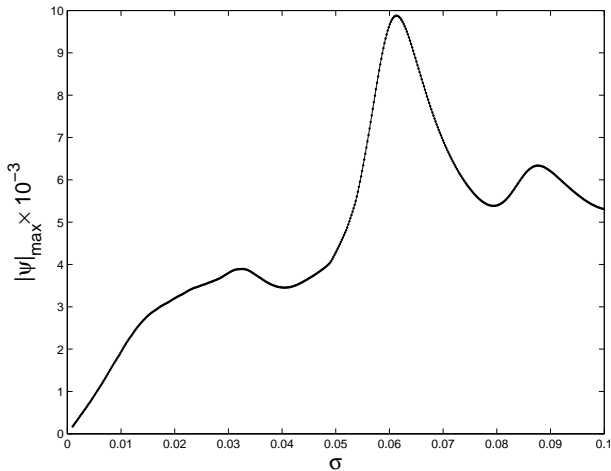
# Rotating tank experiments

- Free modes deep ocean/shallow seas & dispersion relation.



# Rotating tank experiments

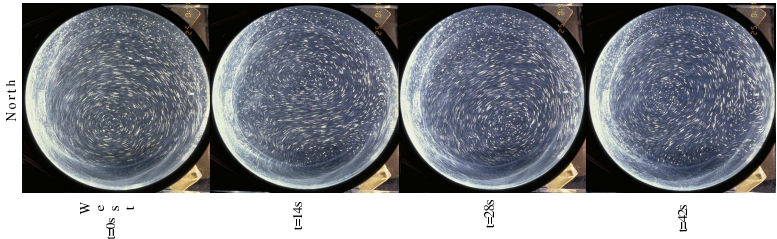
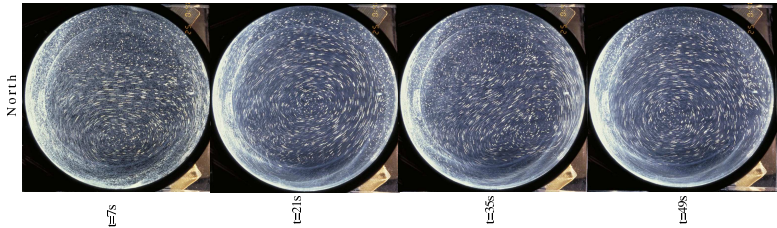
- Results using standard CGFEM of linearized (around rest state) equations. Forcing frequency  $\sigma$ , streamfunction  $\Psi$ .
- **Resonantly excite** hybrid wave modes in ocean basin:



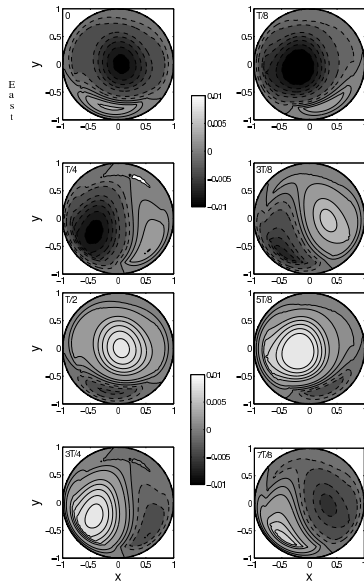
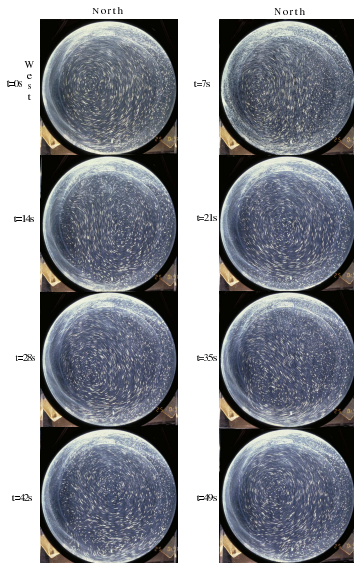
# Rotating tank experiments

- Streak photography  $t = 0, 7, 14, 28, 35, 42, 49$ s forcing period 51.3s ( $\sigma = 0.0612$ ), max. rigid-lid excursion  $\Delta\theta = 2\pi$ ; exposure 2s.

E  
a  
s  
t



# Totating tank experiments: side-by-side



# Outline

- Rotating tank experiments
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## 2D Mathematical Model

General 2D forced-dissipative streamfunction-vorticity model (e.g., Pedlosky, 1987):

$$\frac{1}{A} \frac{\partial \xi}{\partial t} + \nabla \cdot (\mathbf{U} \xi) = \kappa \left( \frac{1}{2} A \omega_T + \alpha A D - \alpha \xi \right), \quad (1)$$

$$\mathbf{U} = \nabla^\perp \Psi, \quad (2)$$

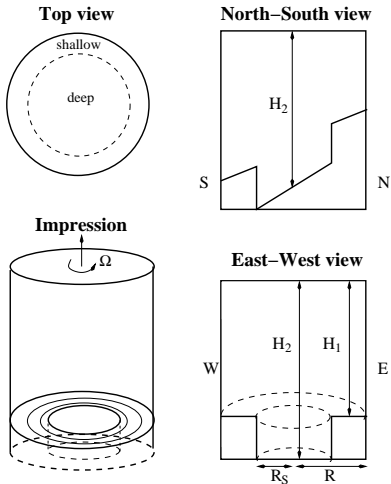
$$\nabla \cdot (A \nabla \Psi) - B \Psi = \xi/A - D, \quad (3)$$

on  $\Omega \subset \mathbb{R}^2$  with:

- (potential) vorticity  $\xi$  (materially conserved),
- streamfunction  $\Psi$ , with velocity  $\vec{u} = A\mathbf{U} = A\nabla^\perp \Psi$ ,
- $\nabla^\perp = [-\partial_y, \partial_x]$ ,
- $0 < A_0 \leq A(x, y) \leq A_1 < \infty$ ,  $B(x, y) \geq 0$  &  $D(x, y) \in \mathbb{R}$ ,
- damping linear in  $\xi$ , term:  $-\kappa\alpha\xi$
- forcing & dissipation cst.  $\kappa$ ,  $\alpha = 1/2, 1$ ; forcing  $\omega_T(x, y, t)$ .

# 2D model: Recall domain for scaling

- 2D Cylindrical domain with depth given "material" function  $H(x, y)$ .



## 2D model: Application to laboratory ocean

- Use  $A = A(x, y) = 1/H(x, y)$
- Fixed ocean depth  $H(x, y)$ ,  $D(x, y) = f_0 = 1$ .

$$H(r, \theta) = \begin{cases} H_{1s} = H_1 - s_1 y & r > R_s + \epsilon \\ H_{2s} + (H_{1s} - H_{2s}) \frac{(r - R_s + \epsilon)}{2\epsilon} & R_s - \epsilon < r < R_s + \epsilon \\ H_{2s} = H_2 - s_2 y & r < R_s - \epsilon \end{cases} \quad (4)$$

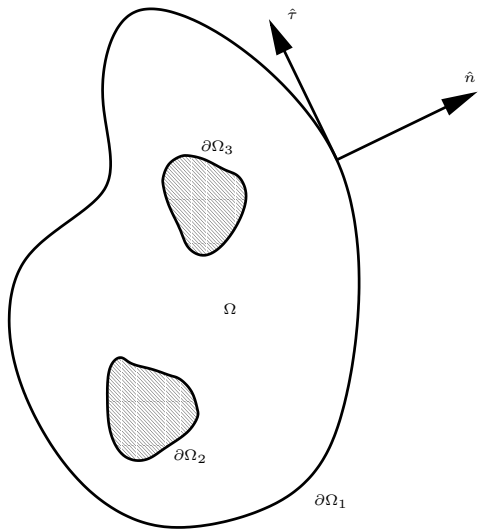
with  $H_1 < H_2$ ,  $s_i = \beta H_i$  and  $\epsilon \ll 1$ , for  $i = 1, 2$ .

- Non-dimensionalized model with (starred=dimensional):

$$t^* = t/f_0, \quad (x^*, y^*) = R(x, y), \quad \Psi^* = \Psi R^2 H_0 f_0, \quad (5)$$

$$\omega^* = f_0 \omega, \quad H^* = H_0 H, \quad f^* = f_0 f \quad \& \quad f_0 = 2\Omega. \quad (6)$$

## 2D model: General domain



## 2D model: Boundary conditions

- On  $\partial\Omega_i$ , we assume **slip flow boundary conditions**

$$\mathbf{U} \cdot \hat{n} = 0. \quad (7)$$

- On such a boundary:

$$0 = \mathbf{U} \cdot \hat{n} = \nabla^\perp \Psi \cdot \hat{n} = \nabla \Psi \cdot \hat{\tau} = \frac{\partial \Psi}{\partial \hat{\tau}}. \quad (8)$$

- Thus  $\Psi$  is independent of  $x$  and  $y$  on such a boundary, but dependent on  $t$ .
- Define  $\Psi|_{\partial\Omega_i}$  implicitly by:

$$\int_{\partial\Omega_i} \mathbf{A} \mathbf{U} \cdot \hat{\tau} d\Gamma = C_i \quad (9)$$

for given constant **circulation**  $C_i$ , since  $dC_i/dt = 0$ .

- **Boundary conditions give energy and enstrophy conservation.**

## 2D model: Conservation laws

In absence forcing/dissipation, generalized vorticity-streamfunction system conserves:

- **energy**,  $dE/dt = 0$ , with

$$E(t) = \frac{1}{2} \|\sqrt{A} \nabla \psi\|_{\Omega}^2 + \frac{1}{2} \|\sqrt{B} \psi\|_{\Omega}^2; \quad (10)$$

- **Casimir invariants**,  $dC/dt = 0$ , with

$$C(t) = \int_{\Omega} (1/A) C_a(\xi) d\Omega; \quad (11)$$

- **enstrophy**,  $dS/dt = 0$ , with

$$S(t) = \frac{1}{2} \|\xi/\sqrt{A}\|_{\Omega}^2. \quad (12)$$

- $L^2$ -norm denoted as  $\|\cdot\|^2 = (\cdot, \cdot)$ ; usual  $L^2$ -inner product.

# Outline

- Rotating tank experiments
- 2D Model
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# CDGFEM: Weak formulation (elliptic equation)

- Define spaces of test functions:

$$W(\Omega) = \left\{ w \in H^1(\Omega) \mid \forall i \exists c \in \mathbb{R} : w|_{\partial\Omega_i} = c \right\} \quad (13)$$

$$W_h^k(\mathcal{T}_h) = W(\Omega) \cap X_h^k(\mathcal{T}_h). \quad (14)$$

- Multiply elliptic equation with arbitrary test function  $w_h \in W_h^k(\mathcal{T}_h)$ ; integrate over domain  $\Omega$ :

$$\int_{\Omega} w_h \nabla \cdot (A \Psi_h) - B \Psi_h w_h + D w_h \, d\Omega = \int_{\Omega} (1/A) w_h \xi_h \, d\Omega. \quad (15)$$

- Integration by parts:

$$\begin{aligned} & \int_{\Omega} A \nabla w_h \cdot \nabla \Psi_h + B w_h \Psi_h \, d\Omega \\ &= \int_{\Omega} D w_h - (1/A) \xi_h w_h \, d\Omega + \int_{\partial\Omega} w_h A \nabla \Psi_h \cdot \hat{n} \, d\Gamma. \end{aligned} \quad (16)$$

# CDGFEM: Weak formulation (elliptic equation)

- Boundary integral in second equation can be written as:

$$\int_{\partial\Omega} w_h A \nabla \Psi_h \cdot \hat{n} \, d\Gamma = \sum_i C_i w_h|_{\partial\Omega_i}. \quad (17)$$

using:

- boundary condition:  $\int_{\partial\Omega_i} A \mathbf{U}_h \cdot \hat{\tau} \, d\Gamma = C_i$ .
- fact that functions  $w_h \in W_h^k$  are constant on each part  $\partial\Omega_i$ .
- Hence, weak formulation of elliptic equation is **symmetric**.

# CDGFEM: Weak formulation (vorticity equation)

- Use space discontinuous test functions for vorticity equation:

$$V_h^k(\mathcal{T}_h) = \left\{ v \mid \forall K \in \mathcal{T}_h \exists w \in X_h^k(\mathcal{T}_h) : w|_K = v|_K \right\}. \quad (18)$$

- Multiply vorticity equation with arbitrary test function  $v_h \in V_h^k(\mathcal{T}_h)$  and integrate over one element  $K \in \mathcal{T}_h$ :

$$\begin{aligned} \int_K (1/A) v_h \frac{\partial \xi_h}{\partial t} dK + \int_K v_h \nabla \cdot (\mathbf{U}_h \xi_h) dK \\ = (v_h, \kappa (A \omega_T / 2 + \alpha A D - \alpha \xi_h))_K. \end{aligned} \quad (19)$$

- Integration by parts:

$$\begin{aligned} \int_K (1/A) v_h \frac{\partial \xi_h}{\partial t} dK - \int_K \xi_h \mathbf{U}_h \cdot \nabla v_h + \int_{\partial K} v_h \xi_h \mathbf{U}_h \cdot \hat{n} d\Gamma \\ = (v_h, \kappa (A \omega_T / 2 + \alpha A D - \alpha \xi_h))_K. \end{aligned}$$

# CDGFEM: Numerical flux

- Consider the boundary integral in the first equation:

$$\int_{\partial K} v_h \xi_h \mathbf{U}_h \cdot \hat{n} \, d\Gamma. \quad (20)$$

- **Problem:**  $\xi_h \in V_h^k(\mathcal{T}_h)$  and  $v_h \in V_h^k(\mathcal{T}_h)$  are not well defined on  $\partial K$  because they can be discontinuous.
- **Solution:**
  - Replace  $v_h$  with  $v_h^-$ , value of  $v_h$  on inside of element  $K$ .
  - Replace  $\xi_h \mathbf{U}_h \cdot \hat{n}$  with *numerical flux*  $\hat{f}(\xi_h^+, \xi_h^-, \mathbf{U}_h \cdot \hat{n})$ , e.g.,
  - Upwind, Central, or Lax-Friedrich fluxes.

# CDGFEM: properties numerical flux

Numerical flux satisfies the following properties:

- (i) **consistency:**

$$\hat{f}(\xi_h, \xi_h, U_n) = \xi_h U_n; \quad (21)$$

- (ii) **conservation:**

$$\hat{f}(\xi_h^+, \xi_h^-, U_n) = -\hat{f}(\xi_h^-, \xi_h^+, -U_n), \quad (22)$$

flux from two neighboring elements  $K_L \in \mathcal{T}_h$  to  $K_R \in \mathcal{T}_h$   
opposite to flux from  $K_R$  to  $K_L$ ;

- (iii)  **$L^2$ -stability in enstrophy norm:**

$$(\xi_h^+ - \xi_h^-) U_n \left( \hat{\xi}_h - \bar{\xi}_h \right) \leq 0 \quad (23)$$

with  $\bar{\xi}_h = (\xi_h^+ + \xi_h^-) / 2$  and  $\hat{\xi}_h = \hat{f}(\xi_h^+, \xi_h^-, U_n) / U_n$ .

# CDGFEM: Numerical flux explicit choices

- Following fluxes  $\hat{f}$  satisfy these properties:

$$\text{central} \quad \hat{f}(\xi^+, \xi^-, U_n) = \frac{\xi^+ + \xi^-}{2} U_n, \quad (24a)$$

$$\text{upwind} \quad \hat{f}(\xi^+, \xi^-, U_n) = U_n \begin{cases} \xi^+ & \text{if } U_n < 0 \\ \xi^- & \text{if } U_n \geq 0 \end{cases}, \quad (24b)$$

$$\text{Lax-Friedrichs} \quad \hat{f}(\xi^+, \xi^-, U_n) = \frac{1}{2} (U_n(\xi^+ + \xi^-) - \alpha(\xi^+ - \xi^-)) \quad (24c)$$

with  $\alpha \geq 0$ .

- Common choice:  $\alpha = \max |U_n|$  with local/global maximum.
- For  $\alpha = 0$  and  $\alpha = |U_n|$ : obtain central and upwind flux.

# CDGFEM: Final weak formulation

- Find  $\xi_h \in V_h^k(\mathcal{T}_h)$  and  $\Psi_h \in W_h^k(\mathcal{T}_h)$  such that for each  $\mathbf{v}_h \in V_h^k(\mathcal{T}_h)$  and for each  $\mathbf{w}_h \in W_h^k(\mathcal{T}_h)$ :

$$\begin{aligned} \int_K (1/A) \mathbf{v}_h \frac{\partial \xi_h}{\partial t} - \xi_h \mathbf{U}_h \cdot \nabla \mathbf{v}_h dK + \int_{\partial K} \mathbf{v}_h^- \hat{f}(\xi_h^+, \xi_h^-, \mathbf{U}_h \cdot \hat{\mathbf{n}}) d\Gamma \\ = (\mathbf{v}_h, \kappa (A \omega_T / 2 + \alpha A D - \alpha \xi_h))_K \end{aligned}$$

and

$$\begin{aligned} \int_{\Omega} A \nabla \mathbf{w}_h \cdot \nabla \Psi_h + B \Psi_h \mathbf{w}_h d\Omega = \sum_i \mathbf{w}_h|_{\partial \Omega_i} \mathcal{C}_i \\ + \int_{\Omega} D \mathbf{w}_h - (1/A) \xi_h \mathbf{w}_h d\Omega. \end{aligned}$$

# CDGFEM: Properties of the scheme

Some properties of the numerical scheme (time *not* discretized):

- **Energy**,  $E_h = \int_{\Omega} A |\nabla \Psi_h|^2 + B |\Psi_h|^2 \, d\Omega$ , is **conserved**.
- **Enstrophy**,  $S_h = \int_{\Omega} (1/A) \xi_h^2 \, d\Omega$ , is **stable** (conserved for central flux).
- **Error estimate**  $\|\xi_h - \xi\|_{L^2(\Omega)} + \|\vec{u}_h - \vec{u}\|_{L^2(\Omega)} < Ch^k$  if  $\xi \in H^{k+1}(\Omega)$ ,  $k > 1$  afor  $k^{\text{th}}$ -order basis functions in finite element space  $X_h^k(\mathcal{T}_h)$ .
- **Explicit 3<sup>rd</sup>-order Runge-Kutta**: (RK3) Shu & Osher (JCP 1988).
- **Implicit modified mid-point rule**: Hairer, Lubich & Wanner (2002).

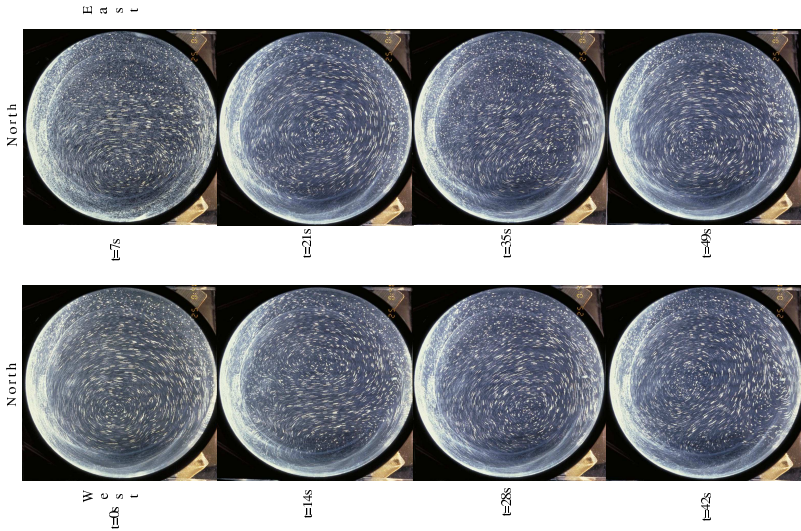
# CDGFEM: Discrete conservation laws

- The space of test functions for the continuous Galerkin discretization must satisfy  $W_h^k \subset V_h^k$ .
- Energy discretized in space and time fully conserved for implicit modified mid-point rule.
- Enstrophy discretized in space and time stable (conserved) for implicit modified mid-point rule with upwind (central) flux.
- For accuracy in case of curved boundaries, isoparametric elements are required.

# Outline

- Rotating tank experiments
- 2D Model
- Continuous-Discontinuous Galerkin Finite Element Method
- **Validation**
- Conclusion

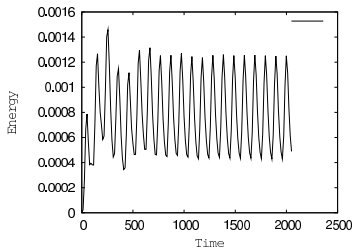
# Validation: recall laboratory measurements



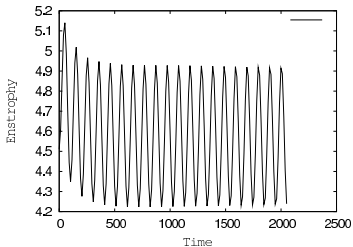
# Validation: quasi-periodic regime

- Spin-up to quasi-periodic state:

(a)



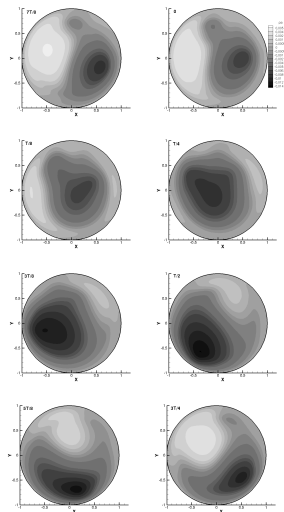
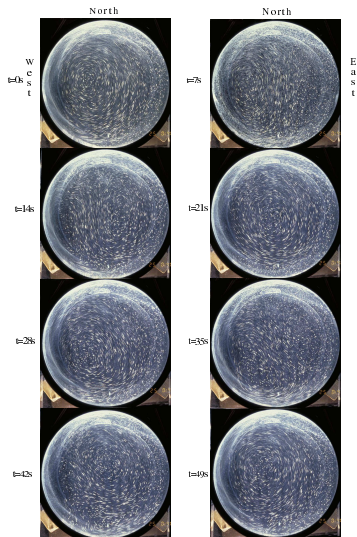
(b)



## Validation: Streamfunction period 19

## Validation: Vorticity period 19

# Validation: side-by-side



# Outline

- Rotating tank experiments
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# Conclusion

- **Used Petsc:** yielded faster and optimized elliptical solver.
- **Reasonable comparison** between laboratory measurements and CDGFEM nonlinear simulations: role model errors?
- **Conservation in time:** Use mid-point rule eith implicit (pseudo-time) solvers.
- **Use actual step-shelf topography** in CDGFEM simulations.
- **Quantify dissipation ratio** in shallow seas and deep ocean; derive effective transparant boundary conditions.
- **Derive effective transparant boundary conditions for deep ocean.**

## Appendix: Error estimate

Van der Vegt, Iszak, B., 2007:

- Detailed error analysis with **minimal smoothness requirements** on vorticity field (extension Liu and Xin, 2000).
- Error in velocity and vorticity **can be estimated as:**

$$\|\xi_h - \xi\|_{0,q,\Omega} + \|\vec{u}_h - \vec{u}\|_{-1,q',\mathcal{T}_h} \leq h^s \|\zeta_0\|_{\Omega} \int_{t_0}^T \|\zeta(\cdot, t)\|_{r,p,\Omega} dt$$

- on a quasi-uniform mesh  $\mathcal{T}_h$  with  **$h < 1$  the element diameter**,  $r > 1/p$ ,  $p > 2$ ,  $1/p < s$ ,  $q = 2p/(p-2)$ ,  $q' = q/(q-1)$ ,  $\min(k-1, r+2/p - \epsilon_0)$ , small  $\epsilon_0 > 0$ .
- $s$  related to degree of differentiability of initial vorticity field  $\zeta_0$  and  $k$  order of polynomial basis functions.

## Validation: Separatrix instabilities

Vorticity: note **chaotic advection** near separatrix.