Numerical Exercises Course Applied Finite Elements 2011

1 Test problem: Laplace equation

Consider Laplace' equation

$$\nabla^2\phi = 0 \qquad \text{on a domain } D: \quad x \in [0, L_x], y \in [0, Ly] \qquad \text{with} \tag{1a}$$

$$\phi(0, y) = 0, \quad \phi(L_x, y) = L_x, \qquad \text{and} \qquad \frac{\partial\phi}{\partial y} = 0 \quad \text{on} \quad y = 0 \text{ and } y = L_y. \tag{1b}$$

1.1 Exercises

1. Check that the exact solution is $\phi = x$.

2. Give the weak formulation: multiply Laplace' equation by a trial or test function, integrate over the domain, use Gauss' theorem and the boundary conditions.

3. Introduce the finite element expansion of ϕ as follows. The numerical approximation $\phi_h = \sum_{k=1}^{N} \phi_j \varphi_j(x, y)$ over the N nodes, where the ϕ_j's are the scalar expansion coefficients and $\varphi_j(x, y)$ the global basis functions with compact support. Choose the test function, which is arbitrary, to be φ_i for all $i = 1, \dots, N$ in turn. Usually, $\varphi_j(x_l, y_l) = 0$ on the nodes (x_l, y_l) if $l \neq j$ and $\varphi_j(x_l, y_l) = 1$ if $l = j$. Substitute this expansion into the weak form to obtain the discretized weak formulation.

4. Express the discretized weak formulation as a linear system $\tilde{A}\,\vec{x} = \vec{b}$ with matrix $\tilde{A}$ and vectors $\vec{x}, \vec{b}$. Determine $\tilde{A}, \vec{x}$ and $\vec{b}$.

5. Define a triangular/quadrilateral mesh on the domain. Make at least three meshes by using refinement. Define global node numbers and the coordinates of each node. Each element has a number. The three/four global node numbers have associated three/four local node numbers. The resulting mesh file consists generally of:

    - an integer expressing the total number of nodes,
    - followed by a list of x and y coordinates of a node,
    - an integer expressing the total number of elements,
    - followed by a list with the global node numbers associated with the four local node numbers,
    - an integer expressing how many boundary nodes there are of a certain type,
    - followed for each boundary type by a list of these global node numbers.

Alternatively, one may mark the boundary nodes with a Dirichlet condition by giving them a negative number or by defining another way to denote the boundary nodes and their type.

6. Define global test and basis functions on this triangular/quadrilateral mesh. Introduce a reference element with a local coordinate system.
7. Identify the local test and basis functions.
8. Describe the global matrix assembly using the local basis functions. This involves a loop over all elements. Clearly indicate how you deal with the Dirichlet and Neumann boundary conditions, and how you build $\tilde{A}$ and $\vec{b}$.
9. Implement your numerical scheme and test it using a linear algebra routine. Ensure that the solutions converge on the different meshes with the use of the L^∞–error and/or L^2-error of ϕ.
10. One may now try another function ϕ as exact solution and construct the corresponding boundary conditions. Use the exact solution for one wave in the next question at time $t = 0$ to define mixed Neuman and Dirichlet boundary conditions. Compare the numerical solution of Laplace' equation with these boundary conditions with this exact solution by determining the L^∞–error and/or L^2-error of ϕ.

2 Linear potential flow for free surface waves

In the three dimensional wave basin at the Maritime Research Institute Netherlands (MARIN), the impact of waves on floating and offshore structures is investigated experimentally. Such laboratory investigations are valuable but expensive. The use of a numerical wave tank is therefore of complimentary value.

We will consider linear waves in a simpler two dimensional wave basin in a vertical cross section for linear potential flow under gravity. The velocity field $\vec{u} = (u, w)^T = \boldsymbol{\nabla}\phi$ is expressed in terms of velocity potential ϕ with gradient operator $\boldsymbol{\nabla} = (\partial/\partial x, \partial/\partial z)^T$ in the horizontal x- and vertical z-direction. The governing equations in $x \in [0, L_x], z \in [-D, 0]$ are

$$\nabla^2\phi = 0 \tag{2a}$$

$$\partial_t\phi + g\,\eta = 0 \qquad \text{at} \qquad \Gamma_S : z = 0 \tag{2b}$$

$$\partial_t\eta = \partial_z\phi \qquad \text{at} \qquad \Gamma_S : z = 0 \tag{2c}$$

$$\hat{\mathbf{n}}_b \cdot \boldsymbol{\nabla}\vec{u} = 0 \qquad \text{at} \qquad \Gamma_B, \Gamma_L \text{ and } \Gamma_R \tag{2d}$$

with free surface deviation η, gravitational acceleration g, $\hat{\mathbf{n}}_b$ the outward normal at the (bottom, left and right) solid boundaries $\Gamma_B : z = -D$ with depth $D > 0$, $\Gamma_R : x = 0$ and $\Gamma_R : x = L_x$, abbreviation $\partial_t = \partial/\partial t$ and so forth. The linearized condition for continuity of pressure at the linearized free surface $\Gamma_S : z = 0$ is given by (2b). The linearized kinematic condition at Γ_S is given (2c). Slip flow at the solid boundary is given by condition (2d) with outward normal $\hat{\mathbf{n}}_b$ at the bottom, left and right boundary Γ_B.

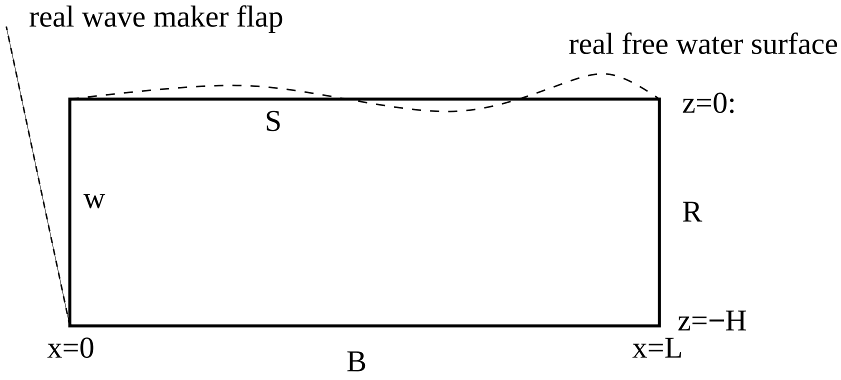

Figure 1: Sketch of the wave basin and its boundaries. $L = L_x$ and $H = D$. The wave maker indicated is not used here.

1. Derive the linear free surface water wave equations including their boundary conditions via the (variational) principle:
$$0 = \delta \mathcal{L}[\phi, \phi_s, \eta] = \int_0^T \int_0^{L_x} \phi_s \partial_t \eta dx dt - \int_0^T \mathcal{H} dt, \tag{3}$$
where $\phi_s(x,t) = \phi(x, z = 0, t)$ and $\mathcal{H}$ is the Hamiltonian (total energy) defined as
$$\mathcal{H} = \int_0^{L_x} \int_D^0 \frac{1}{2} |\nabla \phi|^2 dx dz + \int_0^{L_x} \frac{1}{2} g \eta^2 dx. \tag{4}$$
That is, extremize the functional $\mathcal{L}[\phi, \phi_s, \eta]$ with respect to ϕ, ϕ_s, and η.

2. Derive the finite element discretization by substituting the finite element expansions directly into the above variational principle. Verify that you obtain the same finite element discretization from the standard weak formulation of (2). The expansions of the variables are a sum over global basis functions:
$$\phi_h(x,z,t) = \sum_{j=1}^{N} \varphi_j(x,z)\, \phi_j(t) \tag{5}$$
with global basis function $\varphi_j(x,z)$, and N the number of boundary and interior nodes. At the free surface Γ_S:
$$\phi_{hs}(x,t) = \phi_h(x, z = 0, t) = \sum_{\alpha=1}^{N_s} \varphi_\alpha(x, z = 0)\, \phi_\alpha(t) \tag{6}$$
with α a node on the free surface (most conveniently one take $j = 1, \ldots, N_s$ with $N_s < N$ the number of nodes at this free surface. Likewise $\eta_h(x,t) = \sum_{\alpha=1}^{N_s} \psi_\alpha(x)\, \eta_\alpha(t)$ with $\psi_\alpha(x) = \varphi_\alpha(x, z = 0)$.

3. Subsequently, describe how the integrals can be evaluated using local basis functions. Use linear basisfunctions on triangular/quadrilateral elements.

4. Denote how the global matrices are assembled.

5. Derive or check the following exact solution for free waves in a rectangular basin with $x \in [0, L_x]$ and $z \in [-D, 0]$
$$\phi = \sum_{n=0}^{\infty} \sum_{s=-1}^{1} \big(a_{ns} \cos(\omega_s t) + b_{ns} \sin(\omega_s t)\big) \, cos(\lambda_n x) \cosh\big(\lambda_n (z + H)\big) \tag{7a}$$
$$\lambda_n = \frac{\pi n}{L} \tag{7b}$$
with integer $n > 0$ and frequency
$$\omega_{s=\pm 1} = s \sqrt{g \lambda_n \tanh(\lambda_n H)}. \tag{8}$$
Use for example one mode with one $n \neq 0$ and $s = 1$, but low n, and either a chosen $a_{ns} = 0, b_{ns} \neq 0$, or vice versa, as exact solution for comparison with the numerics. Of course, in principle any linear combination of waves with varying amplitudes a_{ns}, b_{ns} can be chosen.

6. Make a triangular/quadrilateral grid on $x \in [0, L_x]$ and $z \in [-D, 0]$.

7. Use a Crank-Nicholson time integration or using symplectic Euler or symplectic Stormer-Verlet to determine η_α and ϕ_α in time (internet). At least conceptually remove the interior degrees of freedom for ϕ in terms of the surface degrees of freedom for ϕ_s in establishing the time integration scheme.

8. Solve the resulting algebraic system using a linear algebra solver. Take $g = 1$, which will be valid after an appropriate scaling of the equations, and $L_x = 1, D = 1$. Compare the exact and numerical solution by using the energy norm for several mesh refinements, after one and also after 10 wave periods, using sufficient resolution per wave lenght. Show from the exact and numerical solution that the horizontal and vertical resolution does not need to be uniform.

References

Internet.

J. van Kan, A. Segal, and F, Vermolen 2005/2008 *Numerical methods in scientific computing.* VSSD Delft.

J.J.W. van der Vegt and O. Bokhove, reader on finite elements on webpage of Onno Bokhove (google) under his teaching page.