

Question xi (i): Derive linear free surface water wave equations including its boundary conditions by applying an extremizing (variational) principle on the following functional:

$$L_f(\phi, \phi_s, \eta) = \int_0^T \int_{\Gamma_S} \phi_s \partial_t \eta dx dt - \int_0^T H dt, \quad (1)$$

where $\phi_s = \phi(x, z = 0, t)$ and H is the Hamiltonian (total energy) defined as

$$H := \int_{\Omega} \frac{1}{2} |\nabla \phi|^2 dx dz + \int_{\Gamma_S} \frac{1}{2} g \eta^2 dx + \int_{\Gamma_w} V_w \phi dz \quad (2)$$

with $V_w = \partial_t X_w$ the velocity of wave maker.

Derive the finite element discretization by substituting the finite element expansions directly into the above variational principle. Verify that you obtain the same finite element discretization from the standard weak formulation.