Hamiltonian water wave model: balancing dispersion and nonlinearity

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Cotter & B: "Variational water wave model." JEM **67**, 2010
Cooker: "Commemoration of Howell Peregrine." JEM 2010 & Qua Art Qua Science 2010
B & Oliver: "Hamiltonian N-Layer model for atmospheric dynamics." GAFD 2009
Ambati, Van der Vegt & B: "Variational space-time DGFEM free surface waves." Revision JCP
B & Lynch: "Air parcel and air particles: Hamiltonian dynamics." NAW 2007.
B & Oliver: "Parcel Eulerian-Lagrangian fluid dynamics for rotating geophysical flows." Proc. Roy. Soc. 2006

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EGU General Assembley 2010

Outline

- Introduction
- Variational principle for water waves
  - Nonlinear oscillator
  - New model: potential flow water waves with horizontal circulation
- Discontinuous Galerkin finite element models
  - Potential flow water waves
- Conclusions and future work

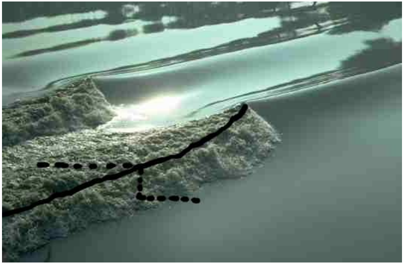
- Depth-averaged shallow water equations workhorse in coastal and marine engineering; shallow means horizontal scales $\gg$ vertical ones:

(a) Prediction of wave run-up/flooding in surf zone. Courtesy: J.-W. Wolf.

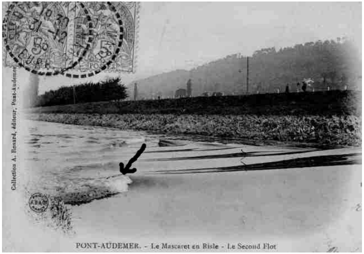
(b) Hydrodynamics and bed erosion. Pettemerzeewering. Courtesy: wiki.

Shallow water equations are simplified with horizontal coordinates x, y and time t:

→ variables are depth $h(x, y, t)$ and depth-averaged horizontal velocity $\mathbf{v}(x, y, t)$
→ bores and hydraulic jumps arise as simplified model of breaking waves . . .



(c) Undular bore in River Severn. Courtesy: D. Howell Peregrine



(d) Mascaret/bore River Risle; breaking/nonbreaking. Courtesy: Malandain '88

- Free surface potential flow water wave model under gravity is widely used for offshore and marine engineering problems:

  → acurate dispersion: (deep-water) waves of different wave length travel with different speeds

  → velocity irrotational $\mathbf{u} = \nabla\phi$

(e) Offshore oil platform "Thunder Horse" tilted by hurricane Dennis.

(f) Wave maker at MARIN, The Netherlands.

Both models approximations of 3D Euler equations with free-surface dynamics.

Advantages:

- inviscid (avoid numerical dissipation wave amplitude)
- SWE: less degrees of freedom due to depth-averaging
- Potential flow: simpler flow field

Disadvantages:

- SWE: no dispersion of waves
- Potential flow: no circulation/vortices, and bores.

Objective 1: Derive a hybrid model with accurate dispersion, bores, and horizontal depth-averaged circulation, via variational/Hamiltonian techniques.

Why variational/Hamiltonian techniques?

Preserve mathematical properties of PDE's and associated conservation laws, energy, mass, wave amplitude . . . .

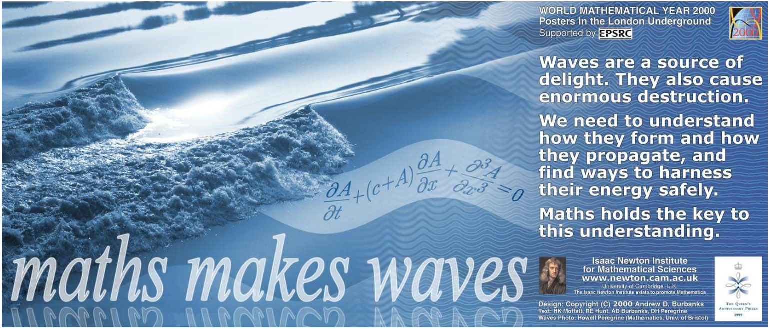
Objective 2: Derive a conservative DGFEM for this new hybrid model to be compatible, no loss of wave amplitude except locally at bores.

Why DGFEM?

Flexibility at moving boundaries, differential geometry, flexibility regarding compatibility: crux lies in flux & function spaces.

Make waves!

with accurate dispersion, no amplitude loss, horizontal circulation and, locally, allow bores as breaking waves.



(g) Photograph: Howell Peregrine; design: Andrew Burbanks

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Ockham: "Frusta fit per plura quod potest fieri per pauciora"

—it is futile to employ many principles when it is possible to employ fewer.

Nonlinear oscillator

- Dynamics of nonlinear ocillator contained in Action Integral

$$0 = \delta L[\chi, \dot{\chi}] := \delta \int_0^T K - V \mathrm{d}t = \delta \int_0^T \frac{1}{2}\dot{\chi}^2 - V(\chi)\mathrm{d}t$$
$$= -\int_0^T \left(\ddot{\chi} + V'(\chi)\right) \delta\chi \mathrm{d}t. \quad (1)$$

position $\chi(t)$; $\dot{\chi} = \mathrm{d}\chi/\mathrm{d}t$; endpoints: $\delta\chi(0) = \delta\chi(T) = 0$.

- One finds therefore:

$$\ddot{\chi} + V'(\chi) = 0. \quad (2)$$

Add constraint:

- Legrendre transform; use Lagrange multiplier λ:

$$0 = \delta\tilde{L}[\chi, \lambda, u] := = \delta \int_0^T \frac{1}{2}u^2 - V(\chi) + \lambda(\dot{\chi} - u)\mathrm{d}t \tag{3}$$

- Variation of u gives: $\delta u : u - \lambda = 0$; substitute $u = \lambda$ back into AI:

$$0 = \delta L^*[\chi, \lambda] := = \delta \int_0^T \lambda\dot{\chi} - (\frac{1}{2}\lambda^2 + V(\chi))\mathrm{d}t \tag{4}$$

- Use similar (Lin) constraints for incompressible water waves in Eulerian case.

Eulerian action principle

- with explicit constraints reads

$$0 = \delta \int_0^T \int_\Omega \frac{1}{2} D |\mathbf{u}|^2 - g\, D\, z + p\,(1 - D) - \underbrace{D\boldsymbol{\pi} \cdot (\partial_t \mathbf{a} + (\mathbf{u} \cdot \nabla)\mathbf{a})}_{\text{labels term}}$$
$$+ \phi\,(\partial_t D + \nabla \cdot (\mathbf{u} D))\,\mathrm{d}\mathbf{x} + \int_{\partial\Omega_s} \lambda(\partial_t h + \mathbf{v}_s \cdot \nabla(h + b) - w_s)\mathrm{d}x\mathrm{d}y\mathrm{d}t \quad (5)$$

- 3D parcel labels $\mathbf{a}(\mathbf{x}, t)$
- Lagrange multipliers $D\boldsymbol{\pi}(\mathbf{x}, t)$ constraining label advection by velocity $\mathbf{u}$
- Lagrange multiplier ϕ constraining density D to satisfy mass continuity
- Lagrange multiplier λ for kinematic free surface condition
- Lagrange multiplier p for incompressibility constraint $D - 1 = 0$.

Strategy

- with explicit constraints:

$$0 = \delta \int_0^T \int_\Omega \frac{1}{2} D|\mathbf{u}|^2 - g\, D\, z + p\,(1 - D) - \underbrace{D\boldsymbol{\pi} \cdot (\partial_t \mathbf{a} + (\mathbf{u} \cdot \nabla)\mathbf{a})}_{\text{labels term}}$$
$$+ \underbrace{\phi\,(\partial_t D + \nabla \cdot (\mathbf{u} D))}_{\text{density}} \,\mathrm{d}\mathbf{x}\mathrm{d}t \qquad (6)$$

- Potential flow: remove labels term; constraint: $\mathbf{u} = \boldsymbol{\nabla}\phi$ (Luke 1967)
- New model: approximate labels term to horizontal/depth-averaged ones

$$\frac{1}{h}\int_b^{h+b} D\,\mathrm{d}z\,\partial_t \mathbf{a} + \frac{1}{h}\int_b^{h+b} D\,\mathbf{u}\,\mathrm{d}z \cdot \boldsymbol{\nabla}\mathbf{a} = 0. \qquad (7)$$

New model with horizontal circulation

- Horizontal labels $\mathbf{a} = \mathbf{a}(x, y, t)$ and multipliers $\boldsymbol{\pi} = \boldsymbol{\pi}(x, y, t)$ are kept

$$
\begin{aligned}
0 =&\delta \int_0^T \int_\Omega \frac{1}{2} D|\mathbf{u}|^2 - gDz + p(1 - D) + \phi\left(D_t + \nabla \cdot (\mathbf{u}D)\right) \\
&- D\,\boldsymbol{\pi} \cdot \left(\partial_t \mathbf{a} + \mathbf{u} \cdot \boldsymbol{\nabla} \mathbf{a}\right) \mathrm{d}\mathbf{x} + \int_{\partial\Omega_s} \lambda(\partial_t h + \mathbf{v}_s \cdot \nabla(h + b) - w_s) \mathrm{d}x\mathrm{d}y\mathrm{d}t
\end{aligned}
$$

- $\mathbf{u}$–variation yields: $\mathbf{u} = \boldsymbol{\nabla}\phi + \mathbf{v} \equiv \nabla\phi + (\boldsymbol{\nabla}\mathbf{a})^T \boldsymbol{\pi}$ with $\mathbf{v} = \mathbf{v}(x, y, t)$
- Substitution . . . .

- Substitute $\mathbf{u} = \boldsymbol{\nabla}\phi + \mathbf{v} \equiv \nabla\phi + (\boldsymbol{\nabla}\mathbf{a})^T\boldsymbol{\pi}$ and $\lambda = D_s\phi_s$ into principle and clean up:

$$
\begin{aligned}
0 =&\delta \int_0^T \int_\Omega D\partial_t\phi + D\,\boldsymbol{\pi}\;\cdot\partial_t\mathbf{a} + \frac{1}{2}D|\nabla\phi + \mathbf{v}|^2 + Dgz + p(D-1)\,\mathrm{d}\mathbf{x}\mathrm{d}t \\
\equiv&\delta \int_0^T \Big(\int_{\Omega_H}\int_b^{b+h} D\partial_t\phi + D\,\boldsymbol{\pi}\;\cdot\partial_t\mathbf{a}\,\mathrm{d}x\mathrm{d}y\mathrm{d}z + \mathcal{H}\Big)\mathrm{d}t
\end{aligned} \tag{8}
$$

- with Hamiltonian/energy $\mathcal{H}$.
- Structure similar to canonical one: $0 = \delta\int_0^T -p\dot{q} + H(p,q)\mathrm{d}t$.
- Variation wrt pairs $\{D,\phi\},\{(\phi)_s,h\},\{D\,\boldsymbol{\pi},\mathbf{a}\}$ ...yields Hamiltonian modification of water wave equations.

- System:

$$\nabla^2\phi + \boldsymbol{\nabla}\cdot\mathbf{v} = 0 \tag{9a}$$

$$(\partial_t\phi)_s + \frac{1}{2}|(\nabla\phi)_s + \mathbf{v}|^2 + g\,(h+b) - \mathbf{v}\cdot\bar{\mathbf{u}} = 0 \tag{9b}$$

$$\partial_t h + \boldsymbol{\nabla}\cdot(h\,\bar{\mathbf{u}}) = 0 \tag{9c}$$

$$\partial_t(h\,\mathbf{v}) + \boldsymbol{\nabla}\cdot(h\bar{\mathbf{u}}\mathbf{v}) + h\mathbf{v}\boldsymbol{\nabla}\bar{\mathbf{u}} = 0 \tag{9d}$$

$$h\bar{\mathbf{u}} = \int_b^{b+h} \boldsymbol{\nabla}_H\phi \mathrm{d}z + h\mathbf{v}. \tag{9e}$$

- Depth-averaged $\mathbf{v} = \mathbf{v}(x, y, t)$ and potential vorticity $q(x, y, t) = \nabla\times\mathbf{v}/h$:

$$(\partial_t + \bar{\mathbf{u}}\cdot\nabla)\,q = 0.$$

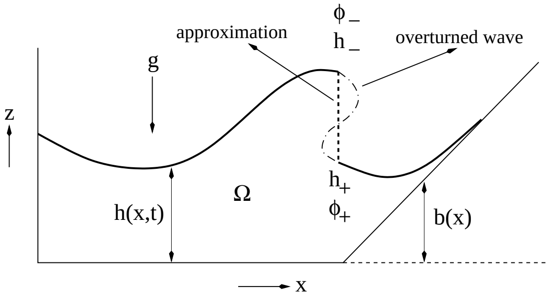
- Potential flow limit: $\mathbf{v} = 0$ or $\mathbf{v} = \boldsymbol{\nabla}_H\varphi$.

- Shallow water limit: $\phi \to \phi_s$, follows from rewritten form

$$\partial_t h + \nabla \cdot (h\,\bar{\mathbf{u}}) = 0 \tag{10a}$$

$$\partial_t(h\bar{\mathbf{u}}) + \nabla_H \cdot (h\,\bar{\mathbf{u}}\,\bar{\mathbf{u}} + \frac{1}{2}g\,h^2) + g\,h\nabla b + h\nabla_H(\partial_t\phi)_s - h\partial_t\left(\overline{\nabla_H\phi}\right) + \frac{1}{2}h\nabla_H(|\mathbf{u}_s|^2 - |\bar{\mathbf{u}}|^2) + h\left(\bar{\mathbf{u}}\,\nabla\overline{\nabla_H\phi} - \bar{\mathbf{u}}\cdot\nabla\overline{\nabla_H\phi}\right) = 0. \tag{10b}$$

- Also leads to weak formulation for bore and hydraulic jumps:



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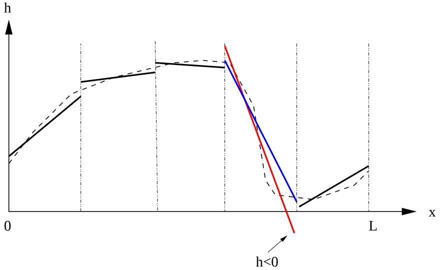
Discontinuous Galerkin finite element models

- Variables represented by continuous functions/polynomials within element K but are discontinuous across



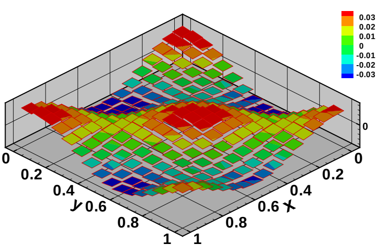
- Discrete variational principle for potential flow case ...

J.C. Luke: "A variational principle for a fluid with a free surface." JFM 1967.

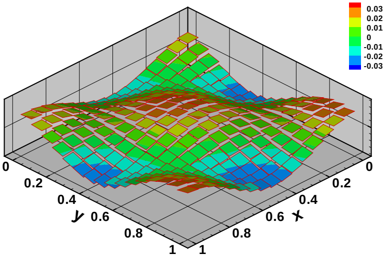
J.W. Miles: "On Hamilton's principle for surface waves." JFM 1977.

Klopman, Van Groesen, Dingemans: "Variational Boussinesq model." JFM 2010

Comparison of variational and standard space-time DG schemes.



(h) Variational space-time DG scheme



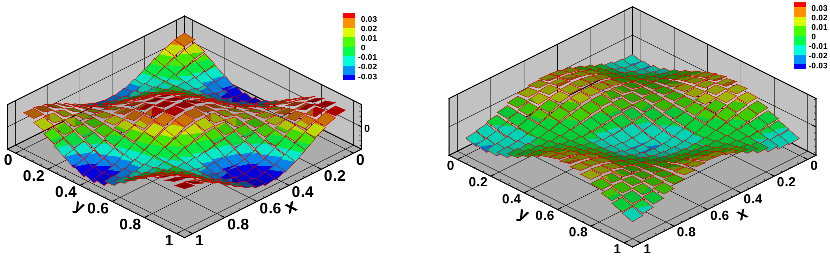
(i) Standard space-time DG scheme

Figure: Free surface evolution at $t = T$ on a grid of size $16 \times 16 \times 4$.

Ambati: Ph.D. Dissertation, University of Twente, eprints.eemcs.utwente.nl, 2008

VdV and Xu: "Space-time DGFEM for nonlinear water waves." JCP 224, 2007.

Comparison of variational and standard space-time DG schemes.

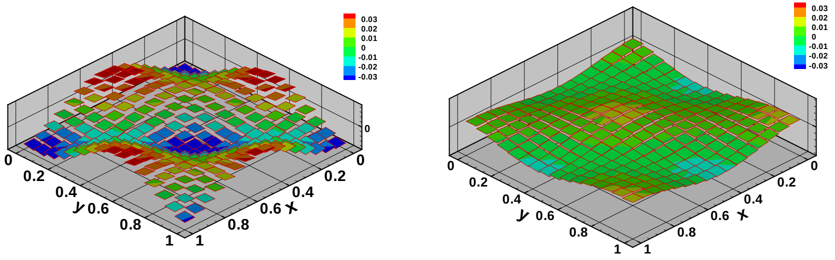


(a) Variational space-time DG scheme

(b) Standard space-time DG scheme

Figure: Free surface evolution at $t = 2T$ on a grid of size $16 \times 16 \times 4$.

Comparison of variational and standard space-time DG schemes.

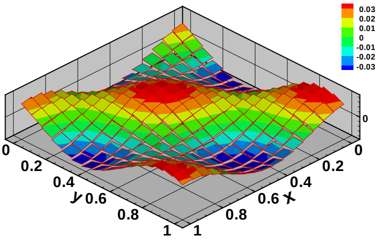


(a) Variational space-time DG scheme

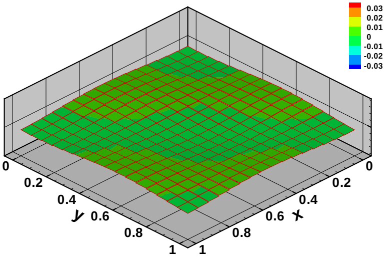
(b) Standard space-time DG scheme

Figure: Free surface evolution at $t = 4T$ on a grid of size $16 \times 16 \times 4$.

Comparison of variational and standard space-time DG schemes.



(a) Variational space-time DG scheme



(b) Standard space-time DG scheme

Figure: Free surface evolution at $t = 8T$ on a grid of size $16 \times 16 \times 4$.

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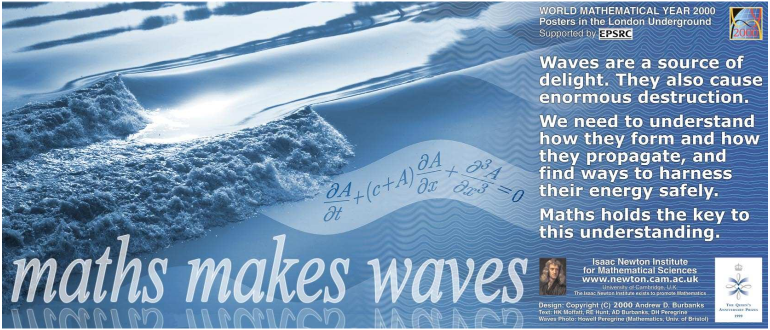
Future work

- Simplification new model in vertical under investigation, with bore relations; explore relation with Klopman et al's Boussinesq model (2010).
- Extension variational DGFEM to nonlinear free surface waves underway.
- Extension to compatible DGFEM for new water water wave model with accurate dispersion and horizontal circulation: NWO project, Ph.D. position.
- **MACE —Maths Assails Coastal Erosion**: fast/accurate flood/storm wave predictions along coast.

Conclusions

- Gone towards goal:



Variational principles: Euler fluid with free surface

Lagrangian action principle incompressible Euler equations with free surface:

- reads, in extension of finite-dimensional case $\chi \to \boldsymbol{\chi}$ & $V(\chi) \to g\chi_3$:

$$0 = \delta L[\boldsymbol{\chi}, \partial_\tau \boldsymbol{\chi}, p] = \delta \int_0^T \int_{\Omega_0} \frac{1}{2} |\partial_\tau \boldsymbol{\chi}|^2 - g\chi_3 + p(1/D - 1)\,\mathrm{d}\mathbf{a}\mathrm{d}\tau \qquad (11)$$

- fluid parcel position: $\mathbf{x} = (x, y, z)^T = \boldsymbol{\chi}(\mathbf{a}, \tau)$ and time $t = \tau$
- continuum of fluid labelled: $\mathbf{a} = (a, b, c)^T = \boldsymbol{\chi}(\mathbf{a}, 0)$
- Jacobian density D: $D^{-1} = \partial(\boldsymbol{\chi})/\partial(\mathbf{a})$; $D\,\mathrm{d}\mathbf{x} = \mathrm{d}\mathbf{a}$
- endpoint conditions: $\delta\boldsymbol{\chi}(\mathbf{a}, 0) = \delta\boldsymbol{\chi}(\mathbf{a}, T) = 0$
- acceleration of gravity g
- Lagrange multiplier and pressure p incompressibility constraint $1/D - 1 = 0$.
- Variations yield:

$$\delta\boldsymbol{\chi} : \frac{\partial^2 \boldsymbol{\chi}}{\partial \tau^2} = -(\nabla p + gz) \circ \boldsymbol{\chi} \quad \text{and} \quad \delta p : D = 1 \qquad (12)$$

- Note that, with $\partial_\tau \boldsymbol{\chi} = \mathbf{u} \circ \boldsymbol{\chi}$:

$$\partial_\tau D^{-1} = D^{-1} \nabla \cdot \mathbf{u} \circ \boldsymbol{\chi} \overbrace{\Rightarrow}^{D=1} \nabla \cdot \mathbf{u} = 0$$

Eulerian action principle, $\chi(\mathbf{a},\tau) \to \mathbf{a}(\mathbf{x},t)$:

- reads

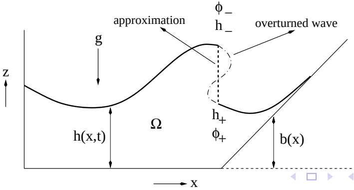
$$0 = \delta L[\mathbf{a}, \partial_t \mathbf{a}, p] = \delta \int_0^T \int_\Omega \frac{1}{2} D|\mathbf{u}|^2 - Dgz + p(1-D)\,\mathrm{d}\mathbf{x}\mathrm{d}t \qquad (13)$$

- velocity $\mathbf{u} = \mathbf{u}(\mathbf{x},t) = (u,v,w)^T$.
- Variations yield incompressible Euler equations, incompressibility condition, and kinetic free-surface condition at S: $z = h + b$, depth $h = h(x,y,t)$ and topography $b = b(x,y)$ with $\mathbf{v}_s$ at S:

$$\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla(p + gz) = 0 \qquad (14a)$$

$$\nabla\cdot\mathbf{u} = 0 \qquad (14b)$$

$$\partial_t h + \mathbf{v}_s\cdot\nabla(b+h) - w_s = 0 \qquad (14c)$$



Luke's variational principle: potential flow water waves

- Variation of

$$0 = \delta \int_0^T \int_\Omega \frac{1}{2} D|\mathbf{u}|^2 - g\,D\,z + p\,(1-D) + \phi\,(\partial_t D + \nabla\cdot(\mathbf{u}D))\,\mathrm{d}\mathbf{x}\mathrm{d}t, \quad (15)$$

- wrt $\mathbf{u}$ yields potential flow: $\mathbf{u} = \boldsymbol{\nabla}\phi$; substitute into variational principle
- to get Luke's variational principle $0 = \delta L[\phi, \phi_s, h]$ (when we set $D = 1$):

$$0 = \delta \int_0^T \iint \int_b^{b+h} \frac{1}{2} D|\nabla\phi|^2 + D\,gz + D\,\partial_t\phi + p(1-D)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z\mathrm{d}t \quad (16)$$

- Variations give water wave equations and Bernoulli:

$$\begin{aligned} &\text{Bernoulli:} & p = \partial_t\phi + \tfrac{1}{2}|\nabla\phi|^2 + gz \\ &\text{3D Laplace:} & \nabla^2\phi = 0 \\ &\text{free surface} & (\partial_t\phi)_s + \tfrac{1}{2}|\nabla\phi|_s^2 + g(h+b) = 0 \\ &\text{dynamics} & \partial_t h + (\boldsymbol{\nabla}\phi)_s\cdot\boldsymbol{\nabla}(h+b) - (\partial_z\phi)_s = 0. \end{aligned} \quad (17)$$