Hamiltonian water wave model -with accurate dispersion and vertical vorticity

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May 2009, Wave-Flow Interactions

Outline

- Introduction
- Variational principle for water waves
  - Nonlinear oscillator
  - Euler equations incompressible fluid with free surface
  - Potential flow water waves
  - New model: potential flow water waves with horizontal circulation
- Discontinuous Galerkin finite element models
  - Depth-averaged shallow water equations
  - Potential flow water waves
- Conclusions and future work

Introduction

- Depth-averaged shallow water equations workhorse in coastal and marine engineering; shallow means horizontal scales $\gg$ vertical ones:

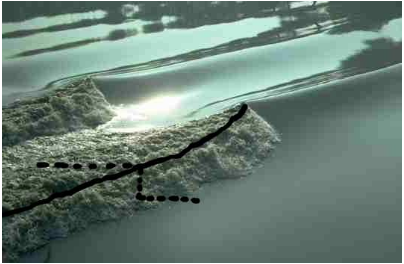
(a) Prediction of wave run-up/flooding in surf zone. Courtesy: J.-W. Wolf.

(b) Hydrodynamics and bed erosion. Pettemerzeewering. Courtesy: wiki.

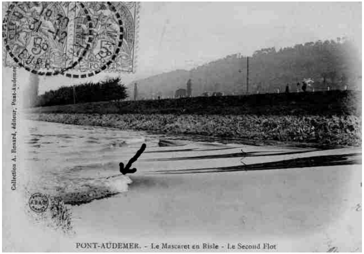
Shallow water equations are simplified with horizontal coordinates x, y and time t:

$\rightarrow$ variables are depth $h(x, y, t)$ and depth-averaged horizontal velocity $\mathbf{v}(x, y, t)$

$\rightarrow$ bores and hydraulic jumps arise as simplified model of breaking waves . . .



(c) Undular bore in River Severn. Courtesy: D. Howell Peregrine.



(d) Mascaret/bore River Risle; breaking/nonbreaking. Courtesy: Malandain '88

- Free surface potential flow water wave model under gravity is widely used for offshore and marine engineering problems:

  → accurate dispersion: (deep-water) waves of different wave length travel with different speeds
  → velocity irrotational $\mathbf{u} = \nabla\phi$

(e) Offshore oil platform "Thunder Horse" tilted by hurricane Dennis.

(f) Wave maker at MARIN, The Netherlands.

Both models approximations of 3D Euler equations with free-surface dynamics.

Advantages:

- inviscid (avoid numerical dissipation wave amplitude)
- SWE: less degrees of freedom due to depth-averaging
- Potential flow: simpler flow field

Disadvantages:

- SWE: no dispersion of waves
- Potential flow: no circulation/vortices, and bores.

Objective 1: Derive a hybrid model with accurate dispersion, bores, and horizontal depth-averaged circulation, via variational/Hamiltonian techniques.

Why variational/Hamiltonian techniques?

Preserve mathematical properties of PDE's and associated conservation laws, energy, mass, wave amplitude . . . .

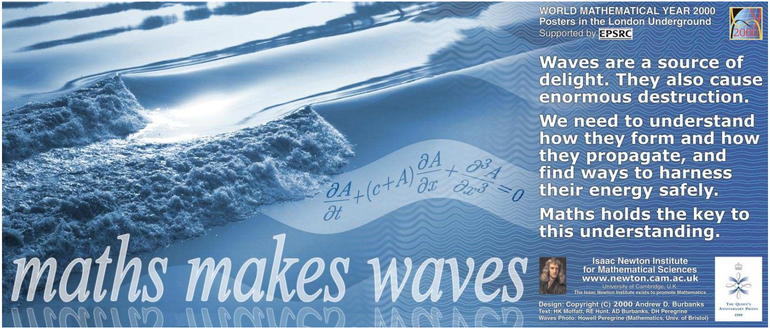
Objective 2: Derive a conservative DGFEM for this new hybrid model to be compatible, no loss of wave amplitude except locally at bores.

Why DGFEM?

Flexibility at moving boundaries, *hp*-adaptivity, flexibility regarding compatibility: the crux lies in the flux and function spaces.

Make waves!

with accurate dispersion, no amplitude loss, horizontal circulation and, locally, allow bores as breaking waves.



(g) Photograph: Howell Peregrine; design: Andrew Burbanks.

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Ockham: "Frusta fit per plura quod potest fieri per pauciora"

—it is futile to employ many principles when it is possible to employ fewer.

Nonlinear oscillator

- Dynamics of nonlinear ocillator contained in action integral

$$0 = \delta L[\phi, \chi] := \delta \int_0^T \phi \dot{\chi} - (\frac{1}{2}\omega^2 \phi^2 + V(\chi)) \mathrm{d}t$$
$$= \delta \int_0^T \phi \dot{\chi} - \underbrace{(K + V)}_{\text{kinetic+potential energy}} \mathrm{d}t \qquad (1)$$

position $\chi(t)$, momentum $\phi(t)$, frequency ω, $\dot{\chi} = \mathrm{d}\chi/\mathrm{d}t$, and endpoint variations $\delta\chi(0) = \delta\chi(T) = 0$.

- Given arbitrariness of $\delta\chi, \delta\phi$:

$$0 = \delta L[\phi,\chi] := \frac{\mathrm{d}}{\mathrm{d}\epsilon} L[\phi+\epsilon\delta\phi, \chi+\epsilon\delta\chi]$$
$$= \int_0^T \delta\phi\dot{\chi} + \phi\delta\dot{\chi} - (\omega^2\phi\delta\phi + V'(\chi)\delta\chi)\mathrm{d}t$$
$$= \int_0^T (\dot{\chi} - \omega^2\phi)\delta\phi - (\dot{\phi} + V'(\chi))\delta\chi\mathrm{d}t \quad (2)$$

- Hamilton's first order equations emerge:

$$\dot{\chi} - \omega^2\phi = 0 \quad \text{and} \quad \dot{\phi} + V'(\chi) = 0. \quad (3)$$

- Combined into one second-order equation derivable from

$$0 = \delta L[\chi,\dot{\chi}] := \delta\int_0^T K - V\mathrm{d}t = \omega^{-2}\delta\int_0^T \frac{1}{2}\dot{\chi}^2 - \omega^2 V(\chi)\mathrm{d}t$$
$$= -\omega^{-2}\int_0^T (\ddot{\chi} + \omega^2 V'(\chi))\delta\chi\mathrm{d}t. \quad (4)$$

Variational principles: Euler fluid with free surface

Lagrangian action principle incompressible Euler equations with free surface:

- reads, in extension of finite-dimensional case $\chi \to \boldsymbol{\chi}$ & $V(\chi) \to g\chi_3$:

$$0 = \delta L[\boldsymbol{\chi}, \partial_\tau \boldsymbol{\chi}, p] = \delta \int_0^T \int_{\Omega_0} \frac{1}{2}(\partial_\tau \boldsymbol{\chi})^2 - g\chi_3 + p(1/D - 1)\,\mathrm{d}\mathbf{a}\mathrm{d}\tau \qquad (5)$$

- fluid parcel position: $\mathbf{x} = (x, y, z)^T = \boldsymbol{\chi}(\mathbf{a}, \tau)$ and time $t = \tau$
- continuum of fluid labelled: $\mathbf{a} = (a, b, c)^T = \boldsymbol{\chi}(\mathbf{a}, 0)$
- Jacobian density D: $D^{-1} = \partial(\boldsymbol{\chi})/\partial(\mathbf{a})$; $D\,\mathrm{d}\mathbf{x} = \mathrm{d}\mathbf{a}$
- endpoint conditions: $\delta\boldsymbol{\chi}(\mathbf{a}, 0) = \delta\boldsymbol{\chi}(\mathbf{a}, T) = 0$
- acceleration of gravity g
- Lagrange multiplier and pressure p incompressibility constraint $1/D - 1 = 0$.
- Variations yield:

$$\delta\boldsymbol{\chi} : \frac{\partial^2 \boldsymbol{\chi}}{\partial \tau^2} = -(\nabla p + gz) \circ \boldsymbol{\chi} \quad \text{and} \quad \delta p : D = 1 \qquad (6)$$

- Note that, with $\partial_\tau \boldsymbol{\chi} = \mathbf{u} \circ \boldsymbol{\chi}$:

$$\partial_\tau D^{-1} = D^{-1} \nabla \cdot \mathbf{u} \circ \boldsymbol{\chi} \overbrace{\Rightarrow}^{D=1} \nabla \cdot \mathbf{u} = 0$$

Eulerian action principle, $\chi(\mathbf{a}, \tau) \to \mathbf{a}(\mathbf{x}, t)$:

- reads

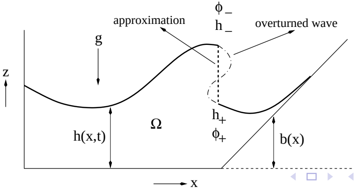
$$0 = \delta L[\mathbf{a}, \partial_t \mathbf{a}, p] = \delta \int_0^T \int_\Omega \frac{1}{2} D \mathbf{u}^2 - Dgz + p(1 - D)\, \mathrm{d}\mathbf{x}\mathrm{d}t \tag{7}$$

- velocity $\mathbf{u} = \mathbf{u}(\mathbf{x}, t) = (u, v, w)^T$.
- Variations yield incompressible Euler equations, incompressibility condition, and kinetic free-surface condition at S: $z = h + b$, depth $h = h(x, y, t)$ and topography $b = b(x, y)$ with $\mathbf{v}_s$ at S:

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla (p + gz) = 0 \tag{8a}$$

$$\nabla \cdot \mathbf{u} = 0 \tag{8b}$$

$$\partial_t h + \mathbf{v}_s \cdot \nabla (b + h) - w_s = 0 \tag{8c}$$



Eulerian action principle

- with explicit constraints reads

$$0 = \delta \int_0^T \int_\Omega \frac{1}{2} D|\mathbf{u}|^2 - g\, D\, z + p\,(1-D) + \underbrace{\boldsymbol{\pi} \cdot (\partial_t \mathbf{a} + (\mathbf{u}\cdot\nabla)\mathbf{a})}_{\text{labels term}} + \phi\,(\partial_t D + \nabla\cdot(\mathbf{u}D))\,\mathrm{d}\mathbf{x}\mathrm{d}t \tag{9}$$

- 3D parcel labels $\mathbf{a}(\mathbf{x}, t)$
- Lagrange multipliers $\boldsymbol{\pi}(\mathbf{x}, t)$ constraining label advection by velocity $\mathbf{u}$
- Lagrange multiplier ϕ constraining density D to satisfy mass continuity
- Lagrange multiplier p for incompressibility constraint $D - 1 = 0$.

Strategy

- with explicit constraints:

$$0 = \delta \int_0^T \int_\Omega \frac{1}{2} D|\mathbf{u}|^2 - g\,D\,z + p\,(1-D) + \underbrace{\boldsymbol{\pi} \cdot (\partial_t \mathbf{a} + (\mathbf{u}\cdot\nabla)\mathbf{a})}_{\text{labels term}} + \underbrace{\phi\,(\partial_t D + \nabla\cdot(\mathbf{u}D))}_{\text{density}} \,\mathrm{d}\mathbf{x}\mathrm{d}t \tag{10}$$

- Potential flow: remove labels term; constraint: $\mathbf{u} = \boldsymbol{\nabla}\phi$
- New model: approximate labels term to horizontal/depth-averaged ones

$$\frac{1}{h}\int_b^{h+b} D\,\mathrm{d}z\,\partial_t \mathbf{a} + \frac{1}{h}\int_b^{h+b} D\,\mathbf{u}\,\mathrm{d}z \cdot \boldsymbol{\nabla}\mathbf{a} = 0. \tag{11}$$

Luke's variational principle: potential flow water waves

- Variation of

$$0 = \delta \int_0^T \int_\Omega \frac{1}{2} D|\mathbf{u}|^2 - g\, D\, z + p\,(1 - D) + \phi\,(\partial_t D + \nabla \cdot (\mathbf{u} D))\, \mathrm{d}\mathbf{x}\mathrm{d}t, \quad (12)$$

- wrt $\mathbf{u}$ yields potential flow: $\mathbf{u} = \boldsymbol{\nabla}\phi$; substitute into variational principle
- and set $D = 1$ to get Luke's variational principle:

$$0 = \delta L[\phi, \phi_s, h] = \delta \int_0^T \iint \int_b^{b+h} \frac{1}{2}|\nabla\phi|^2 + gz + \partial_t\phi\, \mathrm{d}x\mathrm{d}y\mathrm{d}z\mathrm{d}t \qquad (13)$$

- Variations give water wave equations:

$$\begin{aligned} &\text{3D Laplace:} & \nabla^2\phi &= 0 \\ &\text{free surface} & (\partial_t\phi)_s + \tfrac{1}{2}|\nabla\phi|_s^2 + g(h+b) &= 0 \\ &\text{dynamics} & \partial_t h + (\boldsymbol{\nabla}\phi)_s \cdot \boldsymbol{\nabla}(h+b) - (\partial_z\phi)_s &= 0. \end{aligned} \qquad (14)$$

New model with horizontal circulation

- Horizontal labels $\mathbf{a} = \mathbf{a}(x, y, t)$ and multipliers $\boldsymbol{\pi} = \boldsymbol{\pi}(x, y, t)$ are kept

$$
\begin{aligned}
0 =& \delta \int_0^T \int_\Omega \frac{1}{2} D|\mathbf{u}|^2 - gDz + p(1 - D) + \phi\,(D_t + \nabla \cdot (\mathbf{u}D)) \\
& - D\,\boldsymbol{\pi} \cdot \left(\partial_t \mathbf{a} + \mathbf{u} \cdot \boldsymbol{\nabla} \mathbf{a}\right) \mathrm{d}\mathbf{x}\mathrm{d}t
\end{aligned}
\tag{15}
$$

- $\mathbf{u}$–variation yields: $\mathbf{u} = \boldsymbol{\nabla}\phi + \mathbf{v} \equiv \nabla\phi + (\boldsymbol{\nabla}\mathbf{a})^T \boldsymbol{\pi}$ with $\mathbf{v} = \mathbf{v}(x, y, t)$
- Substitution and variation . . . yield modification of water wave equations.

- System:

$$
\begin{aligned}
\nabla^2\phi + \boldsymbol{\nabla}\cdot\mathbf{v} &= 0 && (16a)\\
(\partial_t\phi)_s + \frac{1}{2}|(\nabla\phi)_s + \mathbf{v}|^2 + g\,(h+b) - \mathbf{v}\cdot\bar{\mathbf{u}} &= 0 && (16b)\\
\partial_t h + \boldsymbol{\nabla}\cdot(h\,\bar{\mathbf{u}}) &= 0 && (16c)\\
\partial_t(h\,\mathbf{v}) + \boldsymbol{\nabla}\cdot(h\bar{\mathbf{u}}\mathbf{v}) + h\mathbf{v}\boldsymbol{\nabla}\bar{\mathbf{u}} &= 0 && (16d)\\
h\bar{\mathbf{u}} = \int_b^{b+h} \boldsymbol{\nabla}_H\phi \,\mathrm{d}z + h\mathbf{v}. & && (16e)
\end{aligned}
$$

- Depth-averaged $\mathbf{v} = \mathbf{v}(x, y, t)$ and potential vorticity:

$$(\partial_t + \bar{\mathbf{u}}\cdot\nabla)\,\nabla\times\mathbf{v} = 0.$$

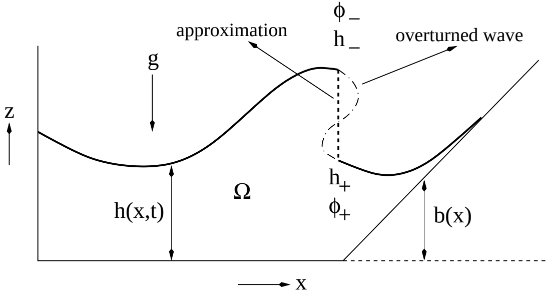
- Potential flow limit: $\mathbf{v} = 0$ or $\mathbf{v} = \boldsymbol{\nabla}_H\varphi$.

- Shallow water limit: $\phi \rightarrow \phi_s$, follows from rewritten form

$$\partial_t h + \nabla \cdot (h \bar{\mathbf{u}}) = 0 \tag{17a}$$

$$\partial_t (h\bar{\mathbf{u}}) + \nabla_H \cdot (h \bar{\mathbf{u}} \bar{\mathbf{u}} + \frac{1}{2} g h^2) + g h \nabla b + h \nabla_H (\partial_t \phi)_s - h\partial_t \left(\overline{\nabla_H \phi}\right) + \frac{1}{2} h \nabla_H (|\mathbf{u}_s|^2 - |\bar{\mathbf{u}}|^2) + h \left(\bar{\mathbf{u}} \nabla \overline{\nabla_H \phi} - \bar{\mathbf{u}} \cdot \nabla \overline{\nabla_H \phi}\right) = 0. \tag{17b}$$

- Also leads to weak formulation for bores and hydraulic jumps:



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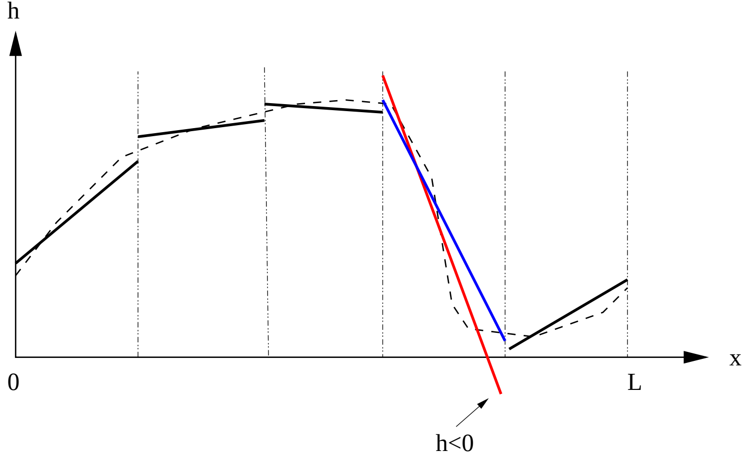
Discontinuous Galerkin finite element models

- Variables are represented by continuous functions/polynomials within an element K but are discontinuous across
- Example, depth field $h = h(x, t)$:



- Space and space-time discretizations.

DGFEM Depth-averaged shallow water equations

- Non-dimensional rotating shallow water equations

$$
\begin{aligned}
\partial_t h + \partial_x(hu) + \partial_y(hv) &= 0\\
\partial_t(hu) + \partial_x(hu^2 + g\,h^2/2) + \partial_y(huv) + g\,h\partial_x b &= f\,h\,v\\
\partial_t(hv) + \partial_x(huv) + \partial_y(hv^2 + g\,h^2/2) + g\,h\partial_y b &= -f\,h\,u\\
\partial_t b &= 0,
\end{aligned}
$$

- water depth h, flow velocity (u, v), topography b are variables of x, y, t
- Coriolis parameter $f = f(x, y)$ given and Froude number $\sqrt{1/g}$ constant.
- Using space DGFEM, expanding h, u, v, and b in above formulation.
- Hyperbolic equations with nonconservative products . . . .

Nonconservative Products

- Rewrite PDE's as system of hyperbolic equations:

$$\partial_t U + G(U)\,\partial_x U = S(U) \tag{18}$$

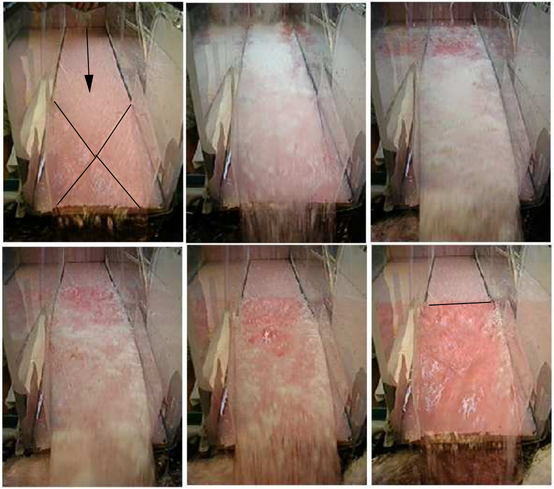
- If there exists $F(U)$ such that $\partial F/\partial U = G(U)$ then nonconservative product $G(U)\partial_x U$ can be written as conservative product: $\partial_x F(U)$.

- Often partial split (e.g. SWE): $G(U)\partial_x U = \partial_x H(u) + \tilde{G}(U)\partial_x U$.

- What if we cannot find a F such that $\partial F/\partial U = G(U)$?
- How do we define $G(U)\partial_x U$ where U is discontinuous?

- This is important because U is discontinuous across every face in the discontinuous Galerkin method.

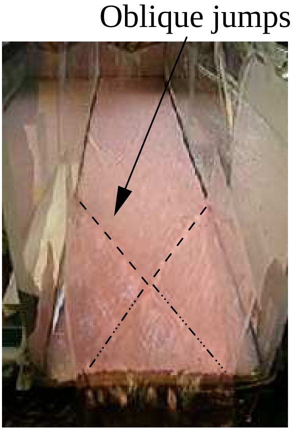
I. Numerical Validation: depth-averaged shallow two-phase flow through contraction

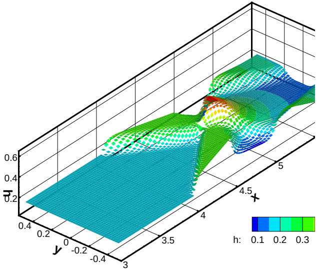


Shallow water flow through a contraction: particles and water

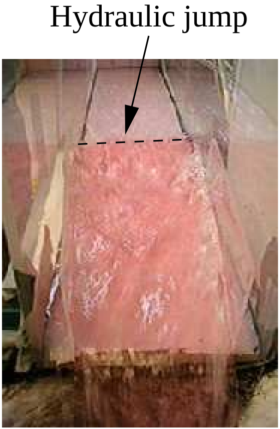
H_2O: theory, experiment and numerics: Akers & B, PoF 2008; numerics Rhebergen *et al.* CMAME 2009

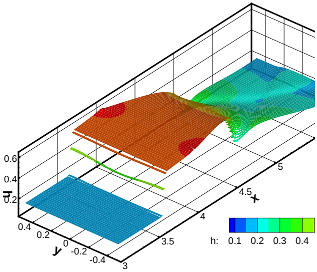
Initial steady-state





Final steady-state





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Luke's variational principle for linear free surface waves

- Domain Ω, $\partial\Omega_S$ mean free surface, $\partial\Omega_L$ lateral boundaries, $\partial\Omega_B$ rigid bed.

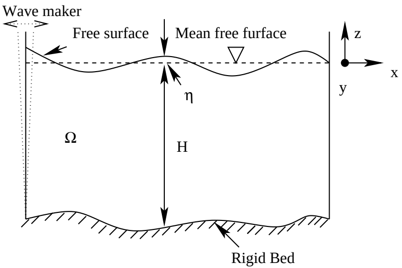


Figure: Sketch of flow domain and its boundaries.

- Velocity field $\mathbf{u}$ irrotational.
- Introduce velocity potential ϕ as $\mathbf{u} = \bar{\nabla}\phi$ such that $\bar{\nabla} \times \mathbf{u} = 0$.

Dynamics of free surface waves (Luke, 1967 and Miles, 1977) contained in following functional, linear case:

$$\mathcal{L}_f(\phi, \phi_s, \eta) := \int_0^T \int_{\partial\Omega_S} \phi_s \partial_t \eta \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}t - \int_0^T \mathcal{H} \, \mathrm{d}t \tag{19}$$

- Hamiltonian:

$$\mathcal{H} := \iint \int_{-H(x,y)}^{0} \frac{1}{2} |\bar{\nabla}\phi|^2 \, \mathrm{d}x\mathrm{d}y\mathrm{d}z + \int_{\partial\Omega_S} \frac{1}{2} g\eta^2 \, \mathrm{d}x \, \mathrm{d}y - \int_{\partial\Omega_L} g_N \phi \, \mathrm{d}(\partial\Omega) \tag{20}$$

- Velocity potential at the mean free surface: $\phi_s = \phi(t, x, y, z = 0)$
- Free surface wave height: η
- Prescribed normal velocity at lateral boundares g_N.

J.C. Luke: "A variational principle for a fluid with a free surface." JFM 1967.

J.W. Miles: "On Hamilton's principle for surface waves." JFM 1977.

Cotter and B: "Water wave model with accurate dispersion and horizontal circulation." Subm. JEM 2009.

- Governing linearized equations:

$$\bar{\nabla}^2\phi = 0 \quad \text{on } \Omega \tag{21}$$

with $\bar{\nabla} = (\partial_x, \partial_y, \partial_z)$ and ϕ the velocity potential.

- Linear free surface boundary conditions:

$$\partial_t\phi + \eta = 0 \quad \text{and} \quad \partial_t\eta - \partial_z\phi = 0 \quad \text{at } \partial\Omega_S \tag{22}$$

with free surface height $\eta(t, x, y)$.

- Eliminating η from (22), we obtain

$$\partial^2_{tt}\phi + \partial_z\phi = 0 \quad \text{at} \quad \partial\Omega_S.$$

Variational space-time DG formulation for linear free surface waves

Basis for space-time formulation

- Tessellate the space-time domain $\mathcal{E}^n$with space-time elements $\mathcal{K}_k^n$

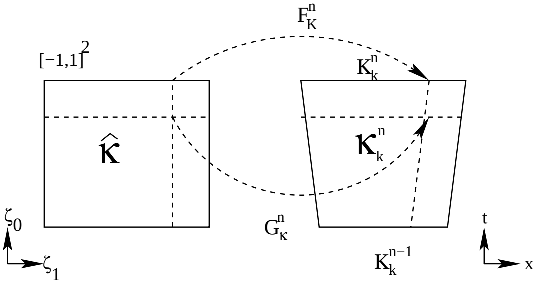


Figure: Geometry of space-time element in computational and physical space.

- Approximations of wave field $(\phi, \mathbf{u}, \eta)$ on each space-time element $\mathcal{K}_k^n$:

$$(\phi_h, \mathbf{u}_h, \eta_h) = (\bar{\phi}_h^n, \bar{\mathbf{u}}_h^n, \bar{\eta}_h^n)\psi^n + (\bar{\phi}_h^{n-1}, \bar{\mathbf{u}}_h^{n-1}, \bar{\eta}_h^{n-1})\psi^{n-1}, \tag{23}$$

  where $(\bar{\phi}_h^n, \bar{\mathbf{u}}_h^n, \bar{\eta}_h^n)$ is approximation of wave field on spatial element K_k^n, and ψ^n and ψ^{n-1} are tent functions in time.

- Global and local lifting operator:

$$\int_{\mathcal{E}_h^n} \mathcal{R}(p) \cdot \tau \, \mathrm{d}\mathcal{K} := \sum_{\mathcal{S}} \int_{\mathcal{S}} p \cdot \{\!\{\tau\}\!\} \, \mathrm{d}\mathcal{S}$$

$$\int_{\mathcal{E}_h^n} \mathcal{R}_{\mathcal{S}}(p) \cdot \tau \, \mathrm{d}\mathcal{K} := \int_{\mathcal{S}} p \cdot \{\!\{\tau\}\!\} \, \mathrm{d}\mathcal{S}. \tag{24}$$

- Discrete functional for linear free surface waves:

$$\mathcal{L}_{\mathrm{h}}(\phi_{\mathrm{h}}, \phi_{\mathrm{h},s}, \eta_{\mathrm{h}}) = \int_{\Gamma_S} \phi_{\mathrm{h},s}(\partial_t \eta_{\mathrm{h}})\, \mathrm{d}\mathcal{S} - \mathcal{H}_{\mathrm{h}}. \tag{25}$$

- Key idea: continuum energy

$$\mathcal{H} = \iint \int_{-H(x,y)}^{0} \frac{1}{2} |\nabla \phi|^2 \mathrm{d}x\mathrm{d}y\mathrm{d}z + \iint \frac{1}{2} g \eta^2 \,\mathrm{d}x\mathrm{d}y \tag{26}$$

- replaced by discrete energy (sum of squares not square of sum):

$$\mathcal{H}_{\mathrm{h}} = \int_{\mathcal{E}_{\mathrm{h}}^n} \frac{1}{2 n_S} \Big(\sum_{\mathcal{S} \subset \partial \mathcal{K}_{\mathrm{k}}^n} \Big| \bar{\nabla} \phi_{\mathrm{h}} + n_S \mathcal{R}_{\mathcal{S},\mathrm{k}}([\![\hat{\phi} - \phi_{\mathrm{h}}]\!]) \Big|^2 \Big) \mathrm{d}\mathcal{K} + \int_{\Gamma_S} \frac{1}{2} g \eta_{\mathrm{h}}^2 \,\mathrm{d}\mathcal{S} - \int_{\Gamma_L} g_N \phi_{\mathrm{h}} \,\mathrm{d}\mathcal{S} \tag{27}$$

Find $\bar{\phi}_h^n \in \bar{V}_h$ and $\bar{\eta}_h^n \in \bar{W}_h$ such that for all $\delta\bar{\phi}_h^n \in \bar{V}_h$ and $\delta\bar{\eta}_h^n \in \bar{W}_h$:

$$\int_{\mathcal{E}_h^n} \bar{\nabla}\phi_h \cdot \bar{\nabla}(\delta\phi_h)\,\mathrm{d}\mathcal{K} - \sum_{\mathcal{S}\in\Gamma_{int}} \int_{\mathcal{S}} \Big(\{\!\{\bar{\nabla}\phi_h\}\!\} \cdot [\![\delta\phi_h]\!] + \{\!\{\bar{\nabla}(\delta\phi_h)\}\!\} \cdot [\![\phi_h]\!] -$$

$$n_S \{\!\{\mathcal{R}_{\mathcal{S}}([\![\phi_h]\!])\}\!\} \cdot [\![\delta\phi_h]\!] \Big)\,\mathrm{d}\mathcal{S} - \int_{\Gamma_L} g_N \delta\phi_h\,\mathrm{d}\mathcal{S} - \int_{\Gamma_S} (\partial_t\eta_h)\delta\phi_{h,s}\,\mathrm{d}\mathcal{S} = 0$$

and

$$\int_{\Gamma_S} \big(\phi_{h,s}\partial_t(\delta\eta_h) - g\eta_h\delta\eta_h\big)\mathrm{d}\mathcal{S} = 0$$

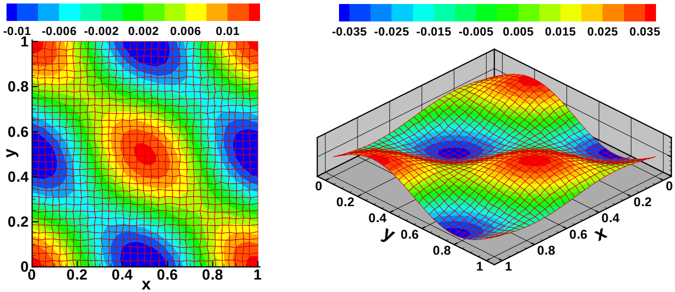
are satisfied with end point conditions

$$\delta\phi_h(t_{n-1}) = \delta\eta_h(t_{n-1}) = \delta\phi_h(t_n) = \delta\eta_h(t_n) = 0. \tag{28}$$

Variations are chosen as: $\delta\phi_h = \psi^n\psi^{n-1}\delta\bar{\phi}_h^n$ and $\delta\eta_h = \psi^n\psi^{n-1}\delta\bar{\eta}_h^n$.

Harmonic Waves

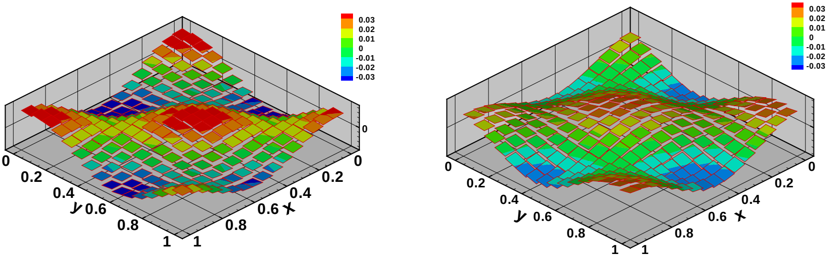
- Two harmonic wave modes: for $p = 1, 2$ verified $2^{\mathrm{nd}}, 3^{\mathrm{rd}}$–order.



(a) Velocity potential. (b) Free surface wave height

Figure: Contour plots of the velocity potential and the free surface wave height at time $t = 0$ on an irregular grid of size $32 \times 32 \times 8$.

Comparison of variational and standard space-time DG schemes.



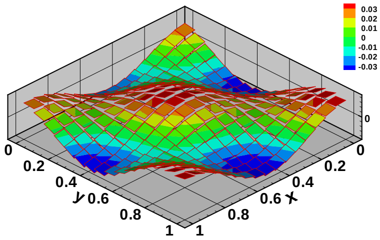
(a) Variational space-time DG scheme (b) Standard space-time DG scheme

Figure: Free surface evolution at $t = T$ on a grid of size $16 \times 16 \times 4$.

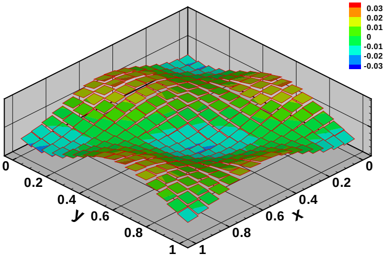
Ambati: Ph.D. Dissertation, University of Twente, eprints.eemcs.utwente.nl, 2008

VdV and Xu: "Space-time DGFEM for nonlinear water waves." JCP 224, 2007.

Comparison of variational and standard space-time DG schemes.



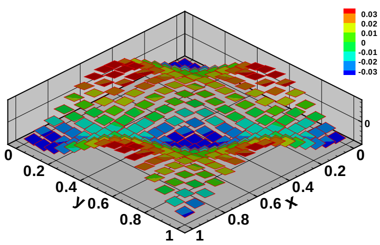
(a) Variational space-time DG scheme



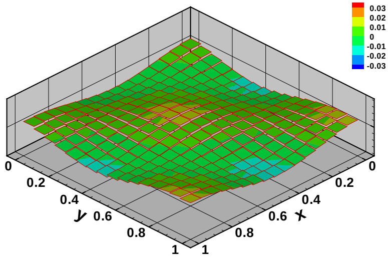
(b) Standard space-time DG scheme

Figure: Free surface evolution at $t = 2T$ on a grid of size $16 \times 16 \times 4$.

Comparison of variational and standard space-time DG schemes.



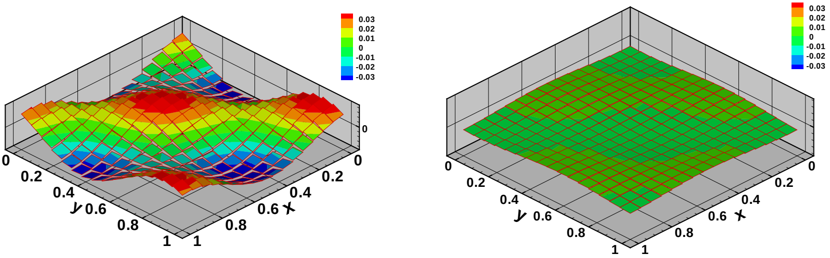
(a) Variational space-time DG scheme



(b) Standard space-time DG scheme

Figure: Free surface evolution at $t = 4T$ on a grid of size $16 \times 16 \times 4$.

Comparison of variational and standard space-time DG schemes.



(a) Variational space-time DG scheme

(b) Standard space-time DG scheme

Figure: Free surface evolution at $t = 8T$ on a grid of size $16 \times 16 \times 4$.

Outline

- Introduction
- Variational principle for water waves
  - Nonlinear oscillator
  - Euler equations with free surface
  - Potential flow water waves
  - Potential flow water waves with horizontal circulation
- Discontinuous Galerkin finite element models
  - Depth-averaged shallow water equations
  - Potential flow water waves
- Conclusions and future work

Future work

- Extension variational DGFEM to nonlinear free surface waves underway.
- Extension to compatible DGFEM for new water water wave model with accurate dispersion and horizontal circulation: NWO project.
- **MACE —Maths Assails Coastal Erosion**: fast/accurate flood/storm wave predictions along coast.

Conclusions

- Novel space-time DGFEM's developed for hyperbolic systems with nonconservative products; applied to rotating shallow water equations.
- Novel variational DGFEM potential flow linear free surface water waves. No decay in amplitude.
- MARIN wave basin: coupled potential flow in deep water to SWE at beach (Klaver '09).