Discontinuous Galerkin finite element methods for hyperbolic nonconservative partial differential equations

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for Sander Rhebergen & Jaap van der Vegt (JCP 227)

Motivation

Equations in conservative form:

$$\partial_t U + \partial_x F(U) = 0.$$

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Equations in nonconservative form where $\partial F / \partial U \neq G(U)$:

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Equations in nonconservative form where $\partial F / \partial U \neq G(U)$:

$$\partial_t U + G(U)\partial_x U = 0.$$

Systems in nonconservative form:

- two-phase flows
  - gas-solid
  - liquid-solid
- shallow water equations

Outline

1. Nonconservative products
2. The Discontinuous Galerkin Finite element method
3. Numerical results
4. Summary

Nonconservative Products

Nonconservative products: $G(U)\partial_x U$.

If there exists an $F(U)$ such that $\partial F/\partial U = G(U)$ then the nonconservative product can be written as a conservative product: $\partial_x F(U)$.

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If there exists an $F(U)$ such that $\partial F/\partial U = G(U)$ then the nonconservative product can be written as a conservative product: $\partial_x F(U)$.

What if we cannot find an F such that $\partial F/\partial U = G(U)$?
How do we define $G(U)\partial_x U$ where U is discontinuous?

This is important because U is discontinuous across every face in the discontinuous Galerkin method.

Solving the problem [a]

Re-define the nonconservative product as:

- where U is continuous:

$$G(U)\partial_x U$$

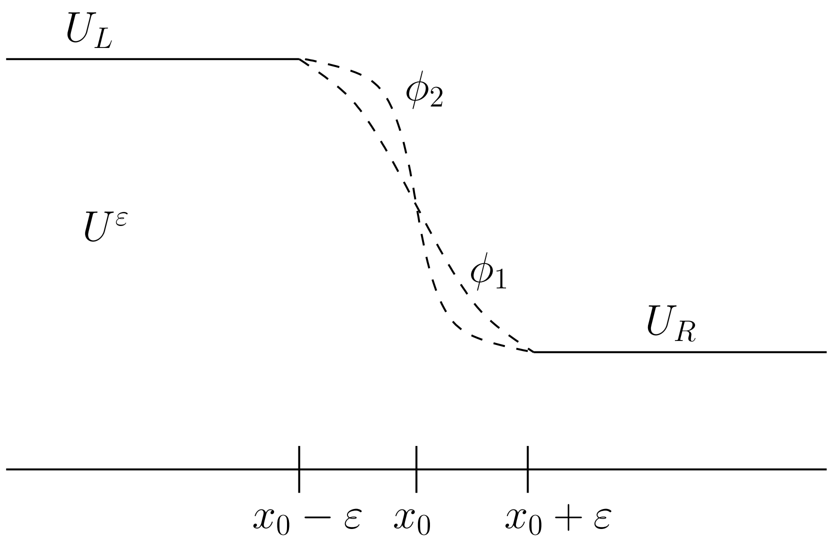
- where U is discontinuous replace U by a continuous U^ε such that:

$$\lim_{\varepsilon \downarrow 0} U^\varepsilon = U.$$

[a] G. Dal Maso, P.G. LeFloch and F. Murat, *J. Math. Pures Appl.*, 1995.

An example

Let U be defined e.g. as:



$$U = \begin{cases} U_L & x < x_0 \\ U_R & x > x_0 \end{cases}$$

Let $U^\varepsilon \to U$ as $\varepsilon \downarrow 0$, e.g.:

$$U^\varepsilon = \begin{cases} U_L & x < x_0 - \varepsilon \\ \phi\left(\frac{x - x_0 + \varepsilon}{2\varepsilon}\right) & x_0 - \varepsilon < x < x_0 + \varepsilon \\ U_R & x > x_0 + \varepsilon \end{cases}$$

Problem solved?

With U^ε defined, we can take the limit of the nonconservative product:

$$\lim_{\varepsilon \downarrow 0} G(U^\varepsilon)\partial_x U^\varepsilon = [G(U)\partial_x U]_\phi, \qquad \text{for } x = x_0$$

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Notation and preliminaries

- The approximated flow domain $\Omega_h \subset \mathbb{R}$ is partitioned into N_n nonoverlapping elements: $\Omega_h = \cup_{k=1}^{N_n} K_k$.
- The boundary of an element K is denoted as ∂K.
- The faces are the intersection of two elements: $\mathcal{S}_{jk} = \overline{K}_j \cap \overline{K}_k$.
- We consider approximations of $U(x,t)$ in the finite element space:

$$W_h = \{V_h \in L^2(\Omega_h) : V|_{K_j} \in \left(P^1(K_j)\right)^d\}$$

with P^1 the space of linear polynomials, $d = \dim(V_h)$ and V_h the polynomial approximation.

Notation and preliminaries (continued)

- The trace of a function $f \in W_h$ at the element boundary ∂K^L is denoted as $f^L = \lim_{\varepsilon \downarrow 0} f(x - \varepsilon n^L)$ with n^L the outward normal at ∂K^L.
- Jump across a face: $[\![f]\!] = f^L n^L + f^R n^R$.
- Average across a face: $\{\!\{f\}\!\} = \frac{1}{2}(f^L + f^R)$.
- Element boundary integrals can be related to face integrals via:

$$\sum_k \int_{\partial K_k} g^L f^L n^L \, d\mathcal{Q} = \sum_{\mathcal{S} \in \mathcal{S}_I} \int_{\mathcal{S}} [\![gf]\!] \, d\mathcal{S} + \sum_{\mathcal{S} \in \mathcal{S}_B} \int_{\mathcal{S}} g^L f^L n^L \, d\mathcal{S}$$

The weak formulation

The weak formulation for equations in nonconservative form, $\partial_t U + G(U)\partial_x U = 0$, is:
Find a $U \in W_h$ *such that for all* $V \in W_h$*:*

$$0 = \sum_j \int_{K_j} V_i(\partial_t U_i + G_{ir}\partial_x U_r)\, dK + \sum_{\mathcal{S}} \int_{\mathcal{S}} [\![V_i]\!] \tilde{H}_i^{nc}\, d\mathcal{S}$$
$$+\sum_{\mathcal{S}} \int_{\mathcal{S}} \{\!\{V_i\}\!\} \left(\int_0^1 G_{ir}(\phi_r(\tau; U^L, U^R)) \frac{\partial \phi_r}{\partial \tau}(\tau; U^L, U^R)\, d\tau \right) n^L\, d\mathcal{S}.$$

$\tilde{H}_i^{nc}$ is the NCP flux which is a stabilizing term. The weak formulation depends on the path ϕ. We take $\phi(\tau; U^L, U^R) = U^L + \tau(U^R - U^L)$.

Design criterium: if $\partial F/\partial U = G(U)$ then weak formulation should reduce to weak formulation for PDE's in conservative form.

Nonconservative form:

$$0 = \sum_j \int_{K_j} V_i(\partial_t U_i + G_{ir}\partial_x U_r)\, dK + \sum_{\mathcal{S}} \int_{\mathcal{S}} [\![V_i]\!] \tilde{H}_i^{nc}\, d\mathcal{S}$$

$$+ \sum_{\mathcal{S}} \int_{\mathcal{S}} \{\!\{V_i\}\!\} \left(\int_0^1 G_{ir}(\phi_r(\tau; U^L, U^R)) \frac{\partial \phi_r}{\partial \tau}(\tau; U^L, U^R)\, d\tau n^L \right) d\mathcal{S}.$$

Conservative form if $\partial F/\partial U = G$, then $G_{ir}\partial_x U_r = \partial_x F_i$:

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Nonconservative form:

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$$+ \sum_{\mathcal{S}} \int_{\mathcal{S}} \{\!\{V_i\}\!\} \left(\int_0^1 G_{ir}(\phi_r(\tau; U^L, U^R)) \frac{\partial \phi_r}{\partial \tau}(\tau; U^L, U^R)\, d\tau n^L \right) d\mathcal{S}.$$

Conservative form if $\partial F/\partial U = G$, then $G_{ir}\partial_x U_r = \partial_x F_i$:

$$0 = \sum_j \int_{K_j} \left(V_i \partial_t U_i - \partial_x V_i F_i\right) dK + \sum_{\mathcal{S}} \int_{\mathcal{S}} [\![V_i]\!] (\{\!\{F_i\}\!\} + \tilde{H}_i^{nc})\, d\mathcal{S}$$

$$\underbrace{+ \sum_{\mathcal{S}} \int_{\mathcal{S}} \{\!\{V_i\}\!\} [\![F_i]\!]\, d\mathcal{S}}_{=0}.$$

Note that $\{\!\{F_i\}\!\} + \tilde{H}_i^{nc} = \widehat{F}_i$ is the numerical flux.

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$$+ \sum_{\mathcal{S}} \int_{\mathcal{S}} \{\!\{V_i\}\!\} \left(\int_0^1 G_{ir}(\phi_r(\tau; U^L, U^R)) \frac{\partial \phi_r}{\partial \tau}(\tau; U^L, U^R)\, d\tau n^L \right) d\mathcal{S}.$$

Conservative form if $\partial F/\partial U = G$, then $G_{ir}\partial_x U_r = \partial_x F_i$:

$$0 = \sum_j \int_{K_j} (V_i \partial_t U_i - \partial_x V_i F_i)\, dK + \sum_{\mathcal{S}} \int_{\mathcal{S}} [\![V_i]\!] (\{\!\{F_i\}\!\} + \tilde{H}_i^{nc})\, d\mathcal{S}$$
$$\underbrace{+ \sum_{\mathcal{S}} \int_{\mathcal{S}} \{\!\{V_i\}\!\} [\![F_i]\!]\, d\mathcal{S} - \sum_{\mathcal{S}} \int_{\mathcal{S}} \{\!\{V_i\}\!\} [\![F_i]\!]\, d\mathcal{S}}_{=0}.$$

Note that $\{\!\{F_i\}\!\} + \tilde{H}_i^{nc} = \widehat{F}_i$ is the numerical flux.

The NCP flux

The NCP flux is given by:

$$\tilde{H}^{nc}(U^L, U^R) = \begin{cases} \frac{1}{2}\int_0^1 G_{ir}(\phi(\tau; U^R, U^L))\frac{\partial\phi}{\partial\tau}(\tau; U^R, U^L)\, d\tau & \text{if} S_L > 0, \\ \frac{1}{2}\big(S_R\bar{U}^* + S_L\bar{U}^* - S_L U^L - S_R U^R\big) & \text{if} S_L < 0 < S_R, \\ \frac{1}{2}\int_0^1 G_{ir}(\phi(\tau; U^L, U^R))\frac{\partial\phi}{\partial\tau}(\tau; U^L, U^R)\, d\tau & \text{if} S_R < 0, \end{cases}$$

where $\bar{U}^*$ is the star-state solution.

- Design criterium: if an F exists such that $\partial F/\partial U = G$ then $\{\!\{F\}\!\} + \tilde{H}^{nc} = F^{HLL}$.
- The derivation of the NCP flux is similar to the derivation of the HLL flux (see Toro, 1997), except that the theory of nonconservative products is used.

Outline

1. Nonconservative products
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Numerical results

We consider the following 1D test cases:

- Dispersed two phase flows
  - subcritical
  - supercritical
- Separated two phase flows in shallow water equations
  - rotating shallow water equations: ocean basins
  - (graded river)

A simplified two phase depth averaged model [a]

Equations for solids phase:

$$\partial_t(\alpha h) + \partial_x(\alpha h u_s) = 0$$

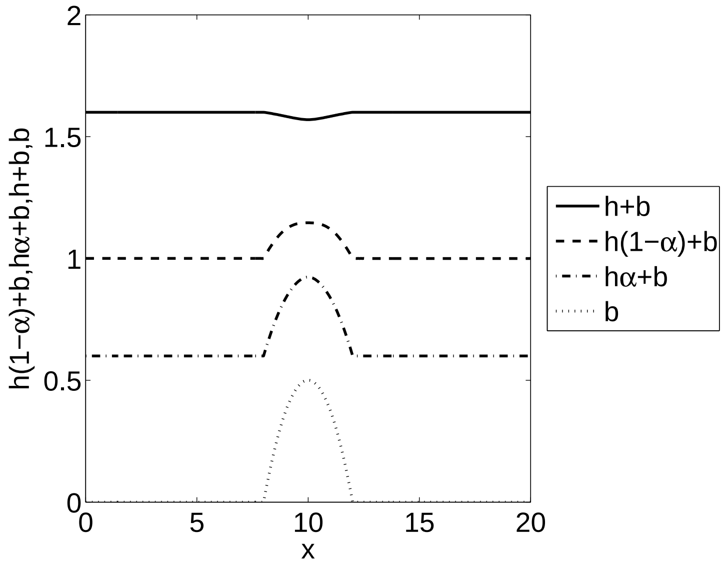
$$\partial_t(\alpha h u_s) + \partial_x\left(\tfrac{1}{2}(1-\rho)\alpha g h^2 + \alpha h u_s^2\right) = -gh\alpha\rho\partial_x h - h\alpha g\partial_x b$$

Equations for liquid phase are similar.

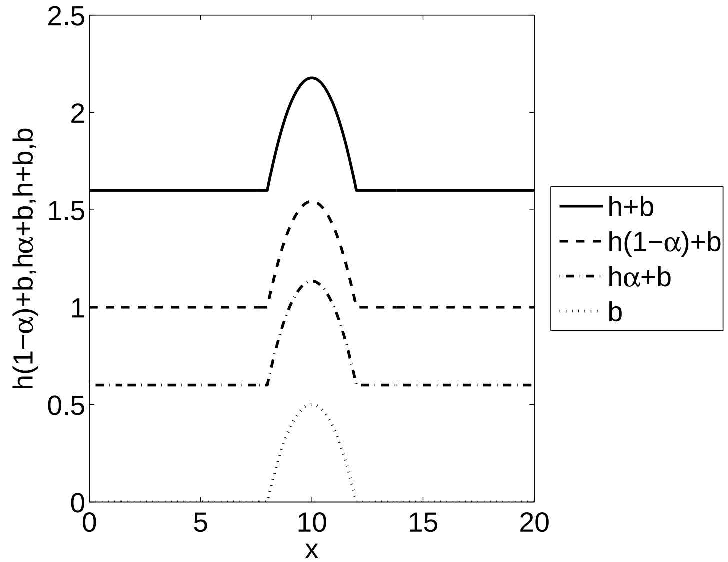
- h depth, α volume fraction of particles, u_f the velocity of the fluid, u_s velocity of particle phase, b topography, ρ ratio between liquid and solids density.
- two sets of shallow water type equations with coupling terms. Coupling terms are nonconservative products.

[a] E.B. Pitman and L. Le, *Phil. Trans. R. Soc. A*, 2005.

Subcritical and supercritical two-phase flow (steady-state)



(a) Subcritical flow.



(b) Supercritical flow.

Convergence subcritical flow

	$h(1-\alpha)+b$				$h\alpha+b$			
N_{cells}	L^2 error	p	$L^{\max}$ error	p	L^2 error	p	$L^{\max}$ error	p
40	$0.8171 \cdot 10^{-3}$	-	$0.2308 \cdot 10^{-2}$	-	$0.1404 \cdot 10^{-2}$	-	$0.4194 \cdot 10^{-2}$	-
80	$0.2025 \cdot 10^{-3}$	2.0	$0.5584 \cdot 10^{-3}$	2.0	$0.3537 \cdot 10^{-3}$	2.0	$0.9903 \cdot 10^{-3}$	2.1
160	$0.4871 \cdot 10^{-4}$	2.1	$0.1322 \cdot 10^{-3}$	2.1	$0.8511 \cdot 10^{-4}$	2.1	$0.2306 \cdot 10^{-3}$	2.1
320	$0.9789 \cdot 10^{-5}$	2.3	$0.2651 \cdot 10^{-4}$	2.3	$0.1712 \cdot 10^{-4}$	2.3	$0.4597 \cdot 10^{-4}$	2.3

Table 1: L^2 and L^{max} error for $h(1-\alpha)+b$, $h\alpha+b$.

Convergence supercritical flow

	$h(1-\alpha)+b$				$h\alpha+b$			
N_{cells}	L^2 error	p	$L^{\max}$ error	p	L^2 error	p	$L^{\max}$ error	p
40	$0.2400 \cdot 10^{-2}$	-	$0.5674 \cdot 10^{-2}$	-	$0.2359 \cdot 10^{-2}$	-	$0.5575 \cdot 10^{-2}$	-
80	$0.6060 \cdot 10^{-3}$	2.0	$0.1402 \cdot 10^{-2}$	2.0	$0.5958 \cdot 10^{-3}$	2.0	$0.1378 \cdot 10^{-2}$	2.0
160	$0.1459 \cdot 10^{-3}$	2.1	$0.3339 \cdot 10^{-3}$	2.1	$0.1434 \cdot 10^{-3}$	2.1	$0.3280 \cdot 10^{-3}$	2.1
320	$0.2933 \cdot 10^{-4}$	2.3	$0.6678 \cdot 10^{-4}$	2.3	$0.2884 \cdot 10^{-4}$	2.3	$0.6561 \cdot 10^{-4}$	2.3

Table 2: L^2 and L^{max} error for $h(1-\alpha)+b$, $h\alpha+b$.

Numerical results

We consider the following 1D test cases:

- Two phase flows
  - subcritical
  - supercritical
- Shallow water equations
  - rotating shallow water equations: ocean basins
  - (graded river)

Rotating shallow water equations: ocean basins

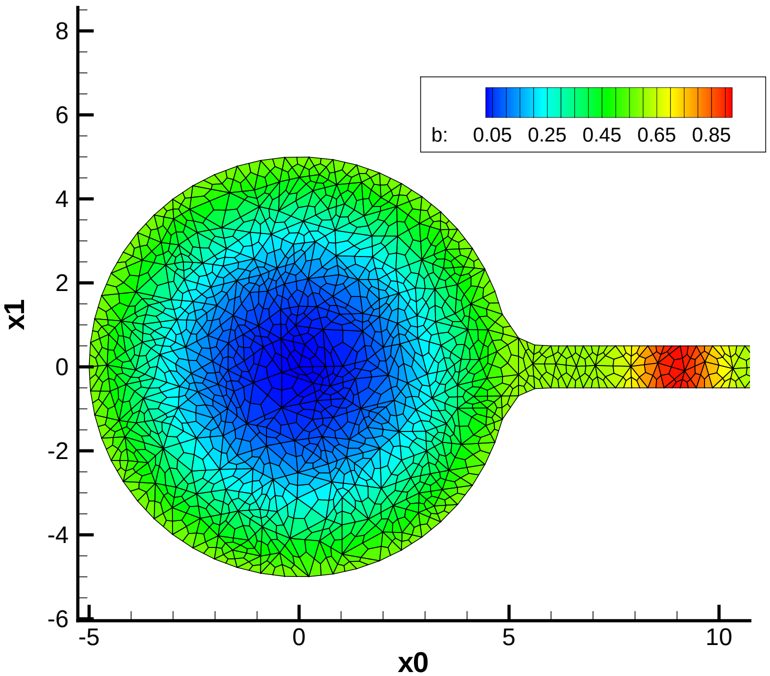
$$\partial_t h + \partial_x(hu) + \partial_y(hv) = 0$$
$$\partial_t(hu) + \partial_x(hu^2 + F^{-2}h^2/2) + \partial_y(huv) - f\,h\,v = -F^{-2}h\partial_x b$$
$$\partial_t(hv) + \partial_x(huv) + \partial_y(hv^2 + F^{-2}h^2/2) + f\,h\,u = -F^{-2}h\partial_y b,$$

where h is water depth, (u, v) flow velocity, b topography β a constant, f Coriolis parameter, and F Froude number.

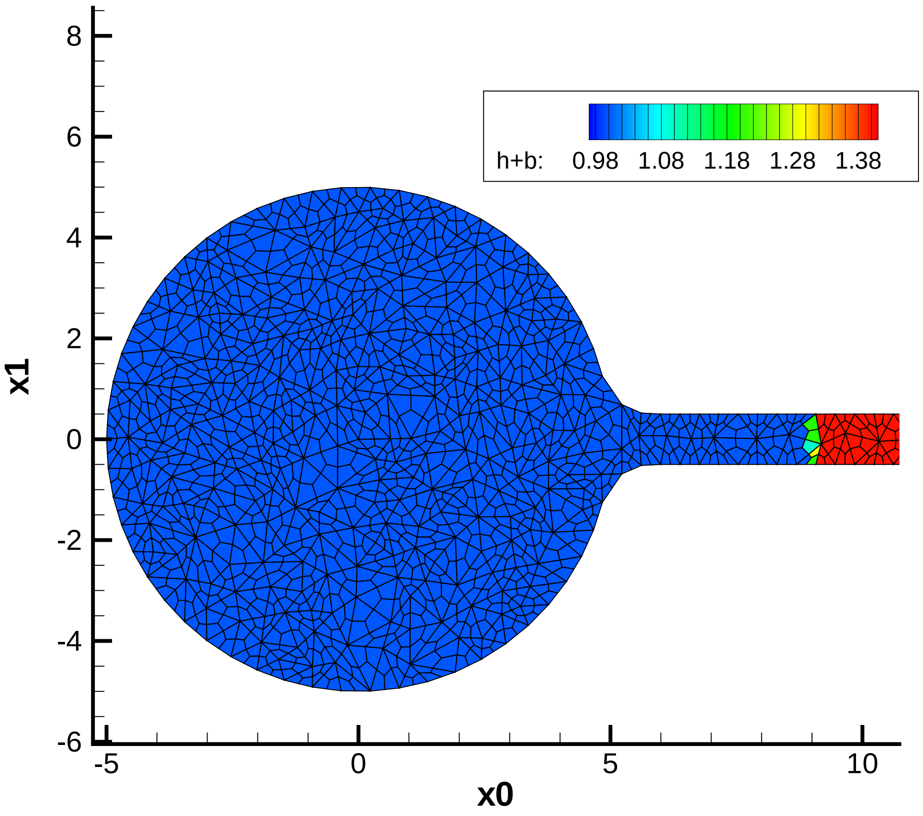
- Using space DGFEM, and expanding h, u, v, and b within the above formulation.
- WENO slope limiter in combination with Krivodonova's discontinuity detector.

Rotating shallow water equations: ocean basins

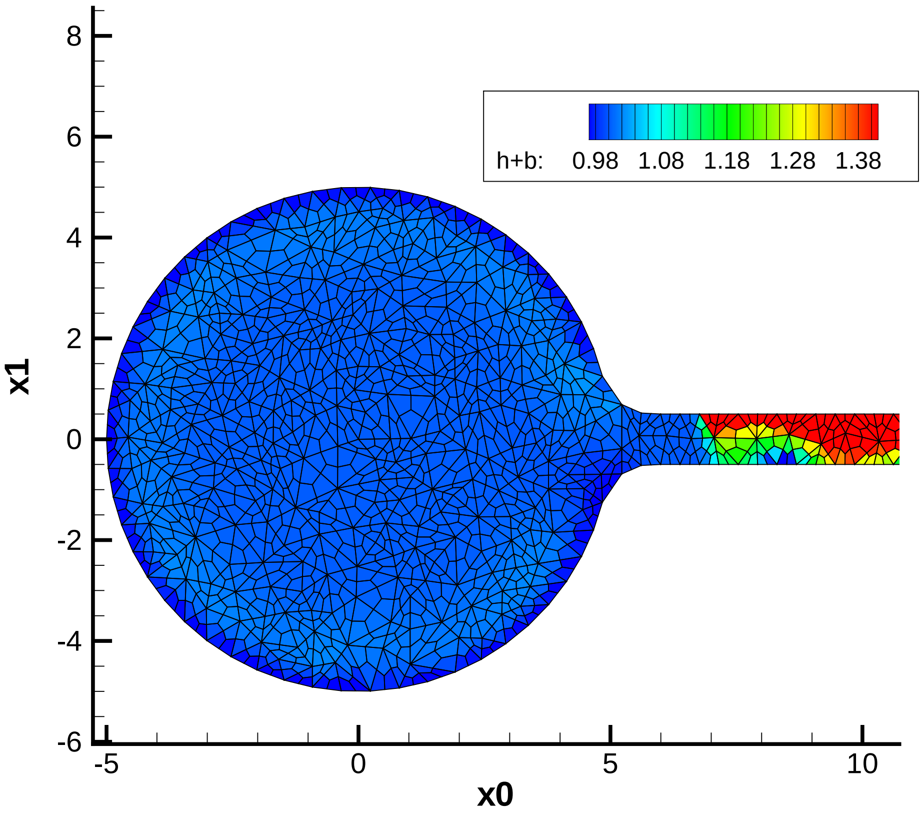
- Deep ocean basin with inflow channel and topography.
- Inflow water: u and h set at end channel (idea K. Helfrich, WHOI).



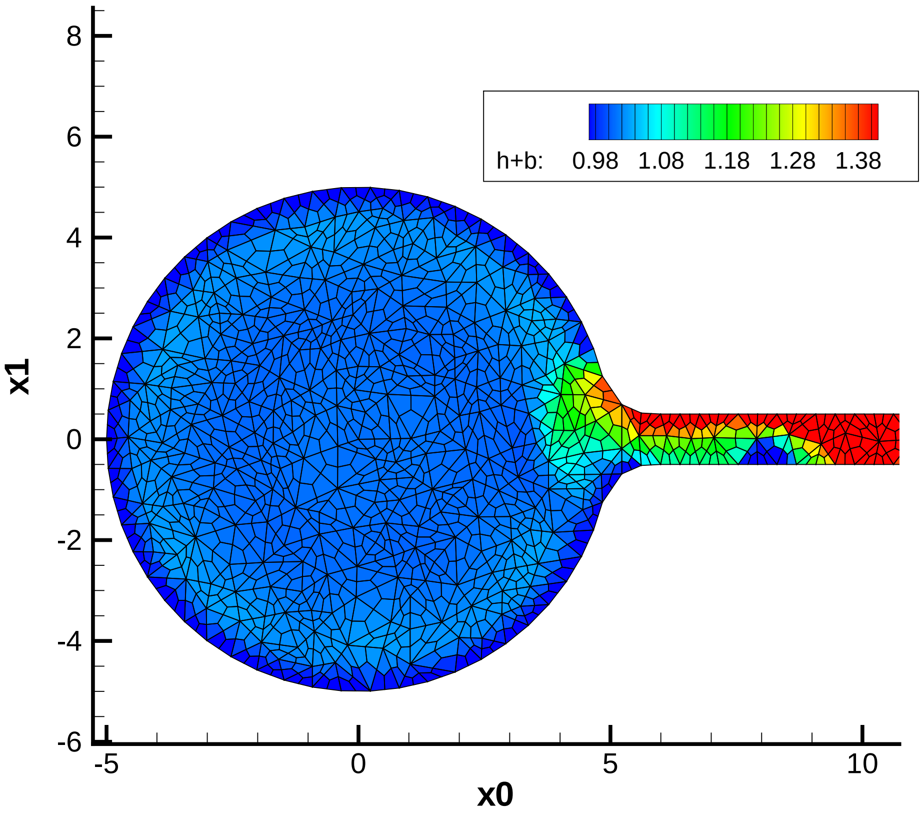
Rotating shallow water equations: $h + b$; $t = 0$



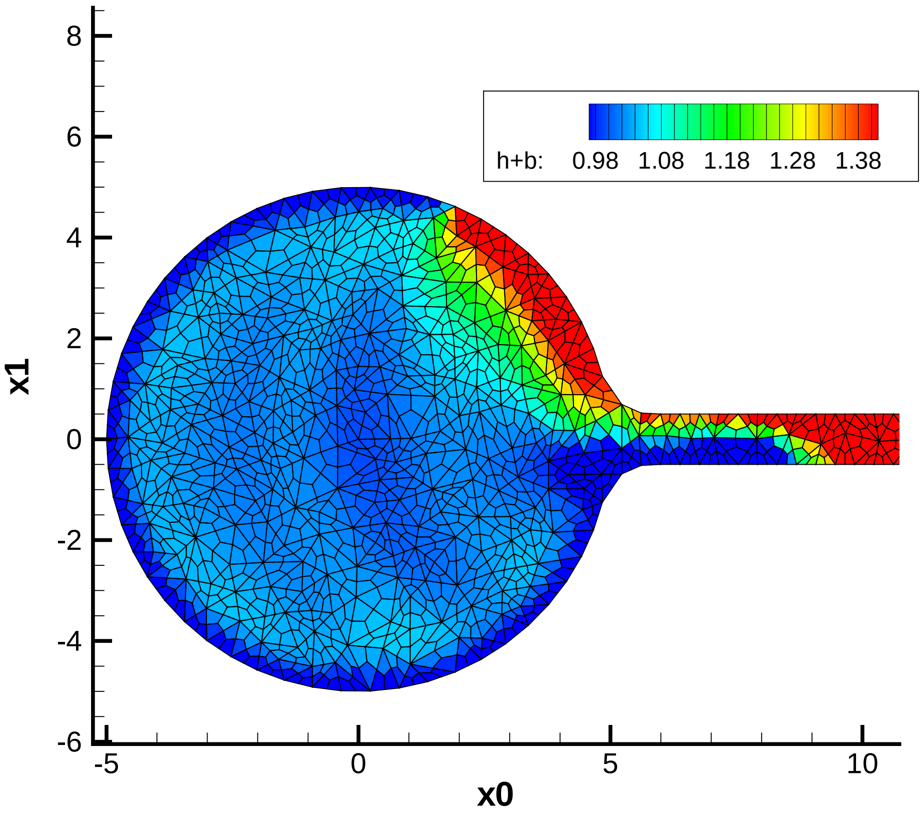
Rotating shallow water equations: $h + b$; $t = 2$



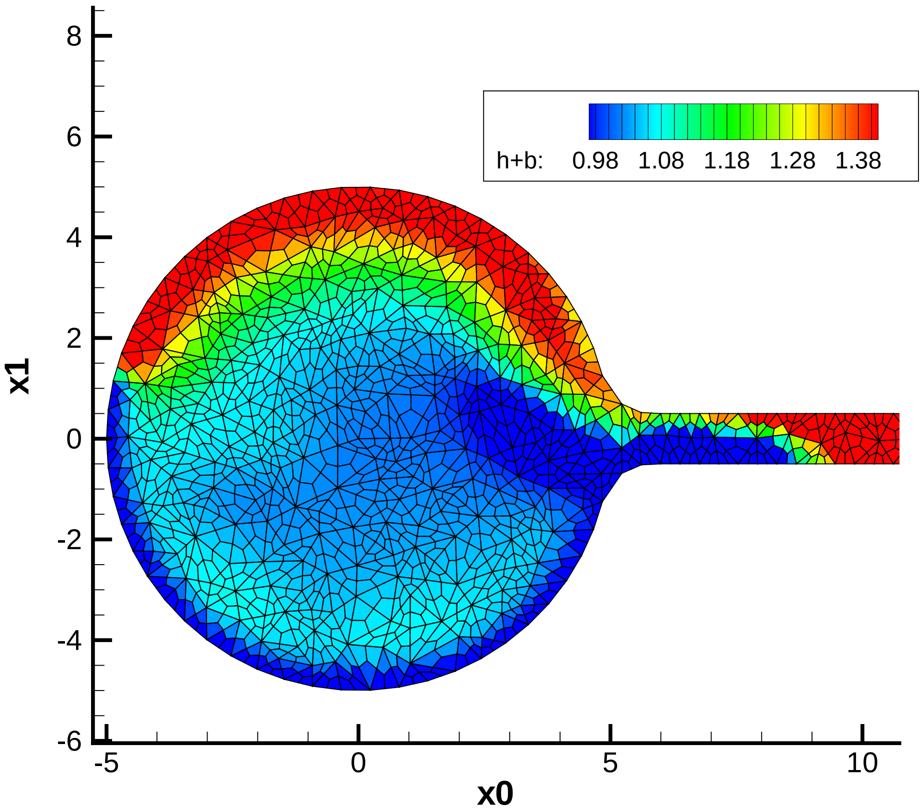
Rotating shallow water equations: $h + b$; $t = 5$



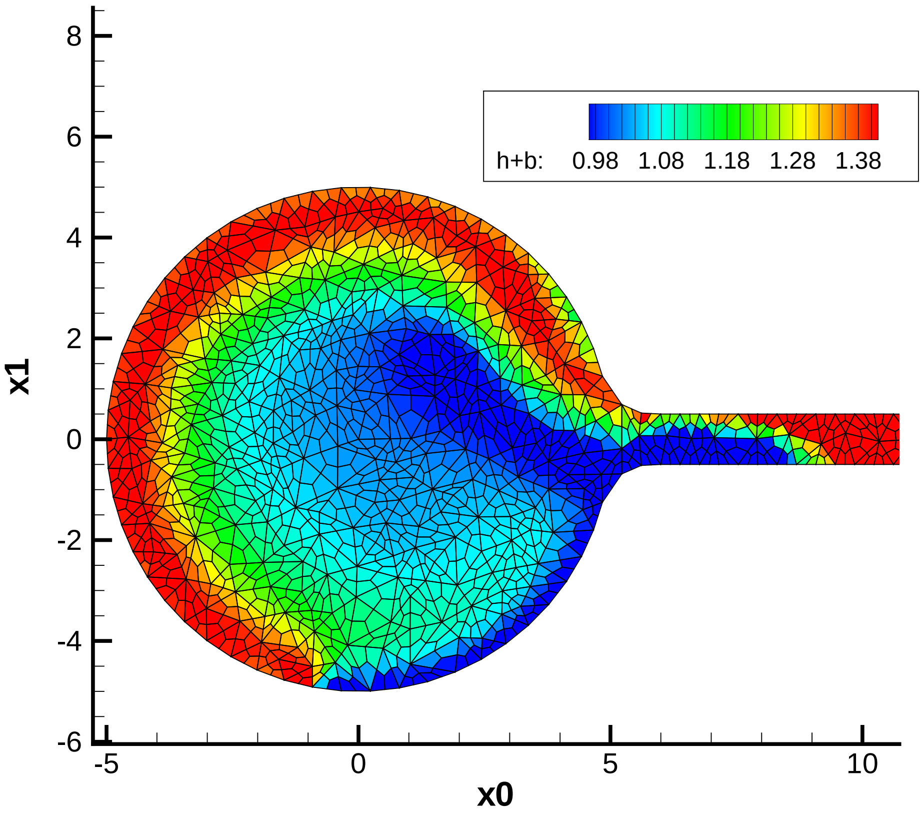
Rotating shallow water equations: $h + b$; $t = 10$



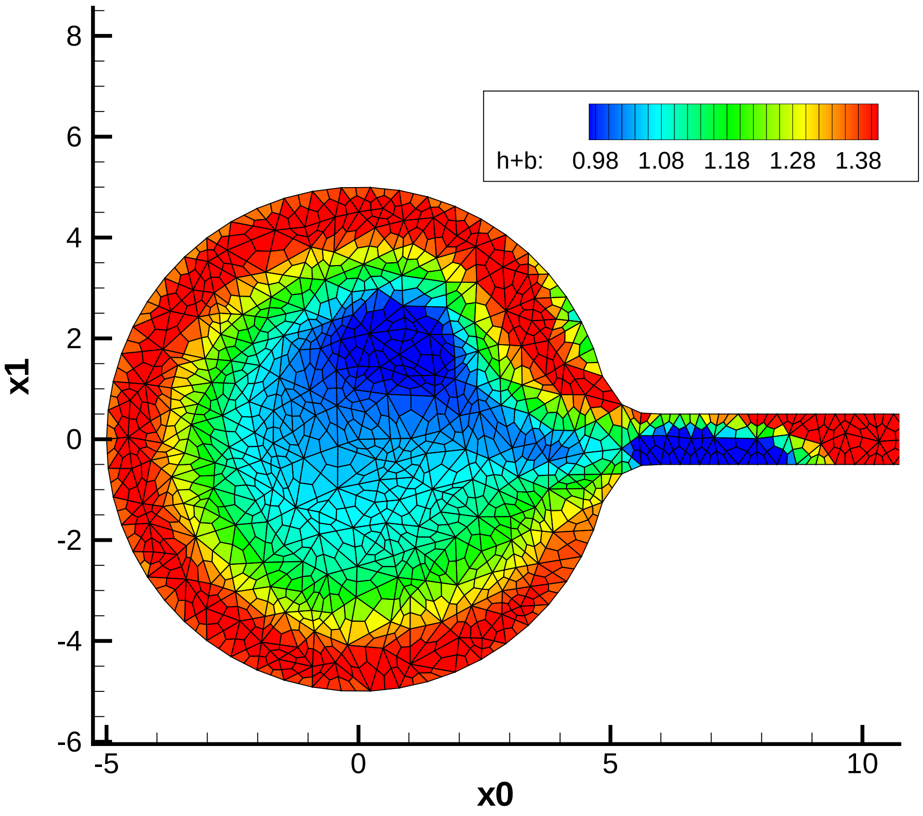
Rotating shallow water equations: $h + b$; $t = 20$



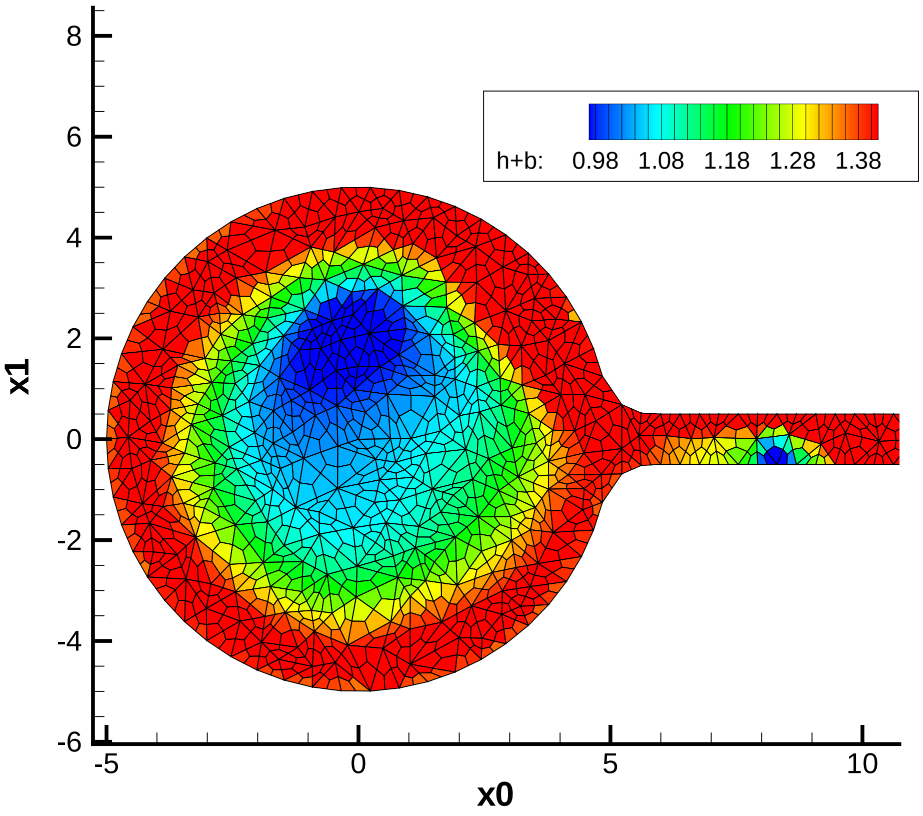
Rotating shallow water equations: $h + b$; $t = 30$



Rotating shallow water equations: $h + b$; $t = 40$



Rotating shallow water equations: $h + b$; $t = 50$



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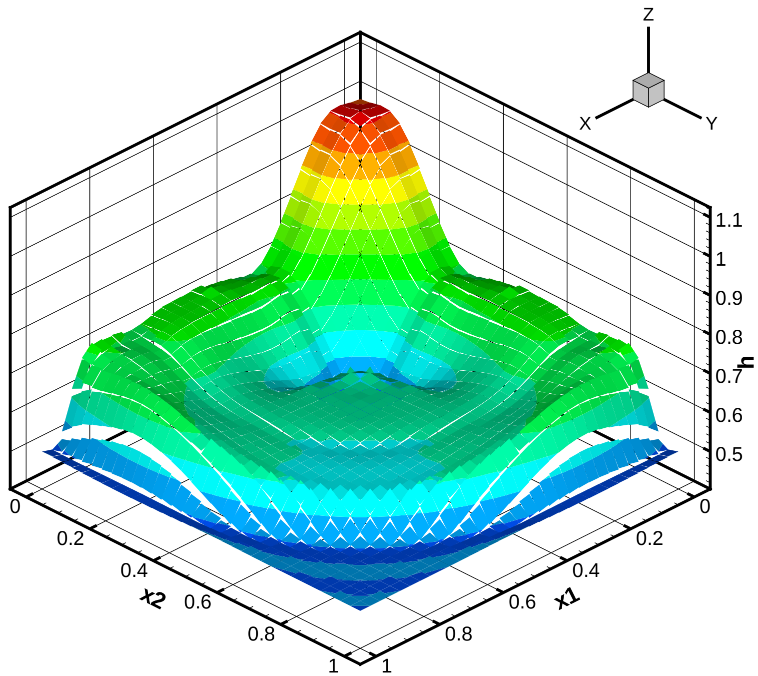
Summary

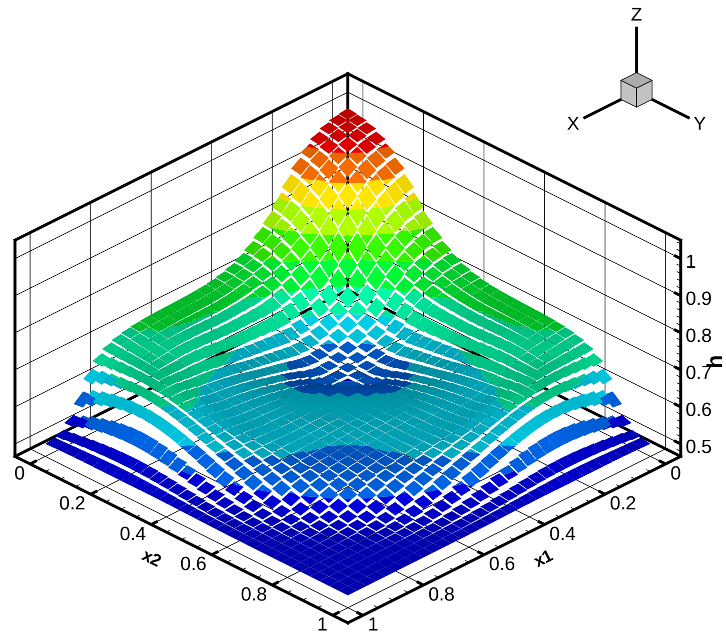
- A discontinuous Galerkin finite element methods for hyperbolic nonconservative partial differential equations.
- A numerical flux (generalization of the HLL flux).
- Second order convergence.
- Shallow ocean flow and multiphase flow applications.
- Shallow flows with 2D bed evolution (morphology)[a]
- For multiphase applications[b]

[a] Pablo Tassi, Wednesday TS310
[b] Sander Rhebergen Tuesday MS140

Future work

- Implementation of the method in multi-dimensions.
- Implementation of Euler-Euler two-phase flow systems.





Articles

- Rhebergen, Bokhove and Van der Vegt 2008: Discontinuous Galerkin finite element methods for hyperbolic nonconservative partial differential equations. *J. Comp. Phys.* 227.
- Tassi, Rhebergen, Vionnet and Bokhove 2008: A discontinuous Galerkin finite element model for morphological evolution under shallow flows. *Comp. Meth. Appl. Mech. Eng.* 197.
- Pesch, Bell, Sollie, Ambati, Bokhove and Van der Vegt 2007: hpGEM- A software framework for Discontinuous Galerkin finite element methods, *ACM Trans. Software* 33.

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