The Holy "Grain" of Breaking Wave Sand Dynamics

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1. Introduction: Cutting Edge

- Mathematical modeling of beach evolution by breaking waves will never be anything less than complicated: large-scale numerical computations are necessary, but because of the large number of (sand) grains involved, as well as the fine-scale water motion, brute-force computations will remain too demanding.
- Creative approaches to computational multi-scale modeling are required in order to reduce the degrees of freedom involved.

Relevance

- Beaches abound world's coastlines.
- Understanding and prediction of beach dynamics lead to insights in order to forecast and prevent the dangers of flooding and erosion.
- Great advances made, e.g., Soulsby (1997), Calantoni et al. (2004, 2006), Garnier et al. (2006, 2008, 2010); Roelvink et al., McCall et al. (Coastal Engineering 2009/2010), Dutykh et al. (2011).
- But laws of sand & sediment transport under breaking waves relatively poorly understood.

The Question

Is it possible to create a manageable mathematical modeling environment of beach dynamics?

- Equations water, air, particle motion in principle available, but DNS too costly.
- The desired modeling environment should permit research on a hierarchy of models.
- Including the range from DNS to depth-averaged models.
- Would such an environment also permit a suitable laboratory experiment for validation?

A Cutting Edge Answer

Yes[2]:

- Imagine we take a giant's knife, make two cuts to isolate a slice of beach . . .
  including sand and water, particle and wave motion.
- Place it between laterally periodic boundaries, or between two glass plates.
- Shrink the latter to table-top size: this Hele-Shaw beach experiment was demonstrated at Fluid Fascinations*url*
- Qua Art & Qua Science show 2010: a tribute to the late Howell Peregrine*url*.

Fluid Fascinations

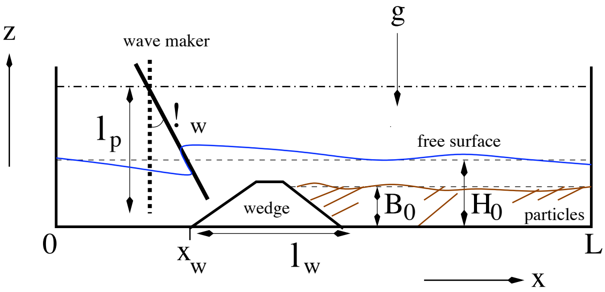
Photo Severn Bore by DHP; painting by zw-artprojects[url]:

2. Mathematical Design of Hele-Shaw Beach

Advantages and disadvantages:

- Set-up unique as it focuses on particle motion by very nonlinear breaking waves.
- Immense reduction of dof's: quasi-2D.
- Innovative: allowing great visualisation & determination fundamental interactions, permitting research on a hierarchy of feasible mathematical modeling.
- Is damping too severe due to the proximity of the glass plates?
- Determine minimal gap distance for which the dynamics is inertial & damping a secondary effect!

Sketch Hele-Shaw beach



Sketch of Hele-Shaw cell with wedge, waterline, particle bottom, and wave-maker rod in two positions.

Specifications

Wave-sand dynamics in a vertical "plane":

- 2 vertical glass plates: $0.6 \times 0.3 \times 2l$ or $1 \times 0.3 \times 2l\,\mathrm{m}^3$
- filled with water & heavier zeolite particles $d \leq 2l\,\mathrm{mm}$
- wavemaker: moving welding rod 1.5mm thin; $f \sim 1\,\mathrm{Hz}$
- Quasi-2D dynamics in Hele-Shaw cell.
- But, what should half gap width l be?

Asymptotic Analysis: width averaging

Determine minimum gap width for which a broken wave can travel from wave maker to beach

- 1. Focus on dynamics water & scale:
  - 3D Navier-Stokes anisotropically with *length & velocity* (L, l, D) & (U, V, W), pressure $P_0 = \rho_0 U^2/(Re\,\epsilon^2)$ scales;
  - dimensionless numbers: Reynolds $Re = UL/\nu$, Froude $1/Fr^2 = gD/U^2$ & aspect ratios $\epsilon = l/L \ll 1$ & $\delta = D/L$:

$$\partial_t w + u\partial_x w + v\partial_y w + w\partial_z w = -\frac{1}{Re\,\epsilon^2\,\delta^2}\partial_z p$$

$$-\frac{1}{Fr^2\delta^2} + \frac{1}{Re}\ \partial_x^2 + \frac{1}{\epsilon^2}\partial_y^2 + \frac{1}{\delta^2}\partial_z^2\ \ w \qquad (1a)$$

$$\partial_x u + \partial_y v + \partial_z w = 0, \qquad (1b)$$

- Simplify:

$$\partial_t w + u\partial_x w + v\partial_y w + w\partial_z w = -\frac{1}{Re\,\epsilon^2\,\delta^2}\partial_z p - \frac{1}{Fr^2\delta^2} + \frac{1}{Re\,\epsilon^2}\partial_y^2 w \quad (2a)$$

$$\partial_x u + \partial_y v + \partial_z w = 0. \quad (2b)$$

- Tempting to assume balance between pressure gradient and dominant viscous terms, dimensionally:

$$u^\star = -\frac{1}{2\nu}(\partial_{x^\star} p^\star/\rho_0)(l^2 - y_\star^2) \quad (3)$$

$$w^\star = -\frac{1}{2\nu}(\partial_{z^\star} p^\star/\rho_0 + g)\,(l^2 - y_\star^2). \quad (4)$$

- 2. Consider ratio inertia terms/pressure gradient:

$$\frac{3\,\rho_0 \tilde{u}^2}{L|\nabla_{xz} p|} = \frac{l^4 |\nabla_{xz} p|}{3\,\rho_0 \nu^2 L} = \frac{l^4 g\,\Delta h}{3\,\nu^2 L^2} \tag{5a}$$

$$= \frac{l^4 \times 10 \times (4 \times 10^{-2})}{3 \times 10^{-12} \times (0.5)^2} \sim 0.1 \text{ to } 10 \tag{5b}$$

  for $l = 0.75$ to $2\,\mathrm{mm}$. Hence, flow is inertial: there is no global Hele-Shaw flow.
- Pohlhausen (Rosenhead 1963) suggested Ansatz:

$$u = \frac{3}{2}\overline{u}\frac{(l^2 - y^2)}{l^2} \qquad \text{and} \qquad w = \frac{3}{2}\overline{w}\frac{(l^2 - y^2)}{l^2} \tag{6}$$

- ...a good approximation, cf. Wilson and Duffy (1998).

- 3. Average across gap: obtain 2D Navier-Stokes equations

$$\partial_t \bar{u} + \gamma \bar{u} \partial_x \bar{u} + \gamma \bar{w} \partial_z \bar{u} = -\frac{1}{\rho_0} \partial_x P - \frac{3\nu \bar{u}}{l^2} \tag{7a}$$

$$\partial_t \bar{w} + \gamma \bar{u} \partial_x \bar{w} + \gamma \bar{w} \partial_z \bar{w} = -\frac{1}{\rho_0} \partial_z P - g - \frac{3\nu \bar{w}}{l^2} \tag{7b}$$

$$\partial_x \bar{u} + \partial_z \bar{w} = 0, \tag{7c}$$

with y–independent pressure $P = P(x, z, t)$, width average $\bar{u} = \ ^{l}_{-l} u(x, y, z, t) \mathrm{d}y/(2l)$ and $\gamma = 6/5$.

Asymptotic Analysis: depth averaging

4. Average in depth –classical– between bottom at $z = b(x,t)$ & free surface at $z = b(x,t) + h(x,t)$.

- Kinematic free surface and bottom boundary conditions:

$$\begin{aligned} \partial_t(b+h) + \bar{\bar{u}}\partial_x(b+h) - \bar{\bar{w}} &= 0 \text{ at } z = h(x,t) + b(x,t) \\ \partial_t b + \bar{\bar{u}}\partial_x b - \bar{\bar{w}} &= 0 \text{ at } z = b(x,t). \end{aligned} \tag{8}$$

- Use hydrostatic balance:

$$\partial_z p/(Re\,\epsilon^2) + 1/Fr^2 = 0 \quad \text{or} \quad \partial_{z^\star} P^\star/\rho_0 + g = 0. \tag{9}$$

Asymptotic Analysis: depth averaging

- Averaging in depth and ignoring (Reynolds) stress terms, gives shallow water equations with linear damping:

$$\partial_t(h\bar{\bar{u}}) + \partial_x \;\gamma h\bar{\bar{u}}^2 + gh^2/2 \; = -\, gh\partial_x b - 3\nu h\bar{\bar{u}}/l^2 \quad (10a)$$

$$\partial_t h + \partial_x(h\bar{\bar{u}}) = 0 \quad (10b)$$

with $h\bar{\bar{u}}(x,t) = \;{}_{b}^{h+b}\, \bar{u}(x,z,t)\mathrm{d}z.$

- Represent breaking wave as shallow-water bore: discontinuity at $x = x_b(t)$ has speed $S = \mathrm{d}x_b/\mathrm{d}t$ satisfying

$$[h(\bar{\bar{u}} - S)] = 0 \quad [h(\bar{\bar{u}} - S)^2 + gh^2/2] = 0. \quad (11)$$

- I performed 1D numerical tests with a fixed beach, and a moving part of bottom as wave maker.

...depth averaging

- Hence, the original design question: Determine the minimum gap width for which a broken wave can travel to from wave maker to beach? leads to the simpler question:
- For which gap width $2l$ can a bore generated by a wave maker reach the end of the beach?
- *Simulations*.

. . . depth averaging

- For which gap width $2l$ can a bore generated by a wave maker reach the end of the beach?
- Answer . . . : for a beach of length ~ 0.5m, the gap width $2\,l > 1.5$mm.
- Given the availability of zeolite particles with $d = 1.80 \pm 0.05$mm, we chose $2\,l = 2$mm
- and simply build two experimental set-ups (B. & Zweers), to test my asymptotic calculations.

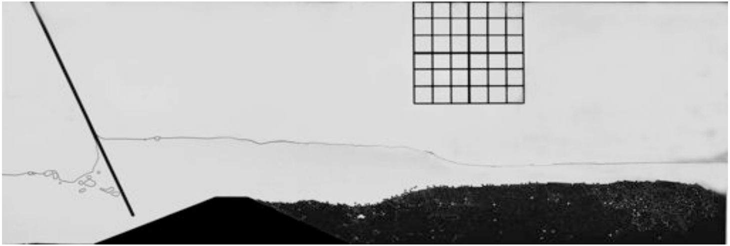
3. Beach and Dune Formation: in the Lab

Wave-sand dynamics in a vertical "plane":

- 2 vertical glass plates: $0.6 \times 0.3 \times 0.002$ or $1 \times 0.3 \times 0.002\text{m}^3$
- filled with water & heavier zeolite particles $d = 1.8$mm
- wavemaker: moving welding rod 1.5mm thin; $f \sim 1\,\text{Hz}$
- "2D" dynamics in Hele-Shaw cell.

Wave-Maker

Wave-maker



Wave Types

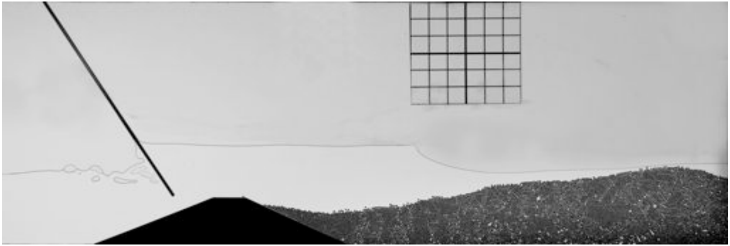
Breaker *wave types* (Peregrine 1983 ARFM) observed:

- Spilling: white water at wave crest spills down front face sometimes with projection of small jet
- Plunging: wave's front face overturns, prominent jet at base wave, causing large splash
- Collapsing: lower portion front face overturns, behaves like truncated plunging breaker
- Surging: significant disturbance smooth profile occurs only near moving shoreline
- ... Shore break: whole face from trough to crest vertical with little/no water in front.

63, 115, 133, 8s

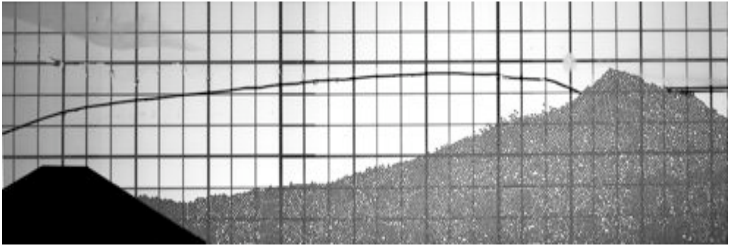
Beach Formation

- Monochromatic wave frequency 0.6m–cell: *beach creation.*
- Monochromatic 0.6hz wave frequency in 1m–cell; 30s stills: *formation of sand waves & beach.*



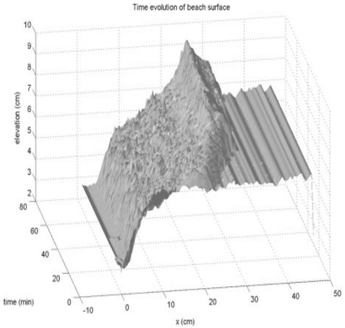
Dune Formation

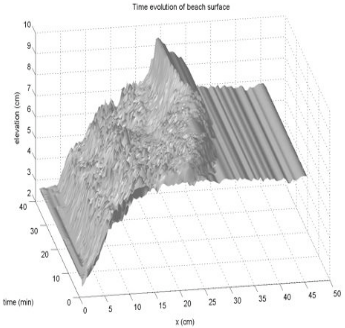
- Alternating $f = 0.6$ & 0.9Hz: *dune creation*, hysteretic effects. Monochromatic $f = 1.0$Hz *dune* (Van der Horn)



Dune Formation: single frequency

- Monochromatic case reproducible (Bram van der Horn):





Modeling challenges

Given design calculations & experimental results, goal becomes:

- to predict the dynamics in the Hele-Shaw cell,
- with models that can be extended to yield feasible 3D-predictions.

Consider the degrees of freedom:

- DNS simulations: $\geq 10^3$ per particle; tank $250 \times 40 \times 1\, d^3 = 10^4\, d^3$.
- Leading to $10^7 - 10^8$ dof's for Navier-Stokes and DPMs! Perhaps feasible in quasi-2D but not in 3D.
- A hierarchy of reduced, multi-scale models is required.

4. Numerical Modeling Expertise

Overview:

- Compatible numerical discretization of variational principle for nonlinear, potential flow, water waves
- Discontinuous Galerkin FEM of shallow water equations: bores, flooding & drying.
- Hamiltonian wave-current model with potential flow & shallow water equations as limits —new.
- Regular extreme-wave: the Bore Solition Splash!

4.1 Compatible discretization water waves

Direct discretization of Miles' variational principle (B. & Ambati).

- Classical, space-time DGFEM for water waves (2007) did not work [well].
- FEM in space and discontinuous Galerkin FEM in time.
- Wave amplitude "preserved".
- Symplectic mesh movement stable due to time integrator: obeys geometric conservation law.
- Numerical verification: *Fenton waves*

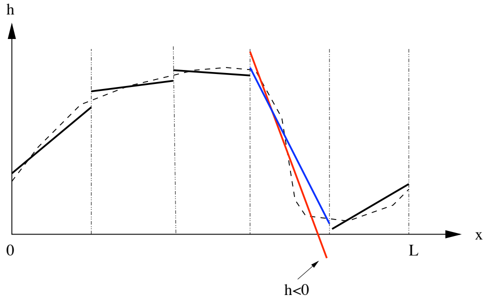
Potential Flow Water Waves

- Numerical validation: *MARIN test case* including *Wave maker movement*.
- Missing: 3rd order in space & 2nd order in time & DG.
- Proof of convergence in progress (Iszak & Van der Vegt).

4.2 DGFEM Shallow Water Equations

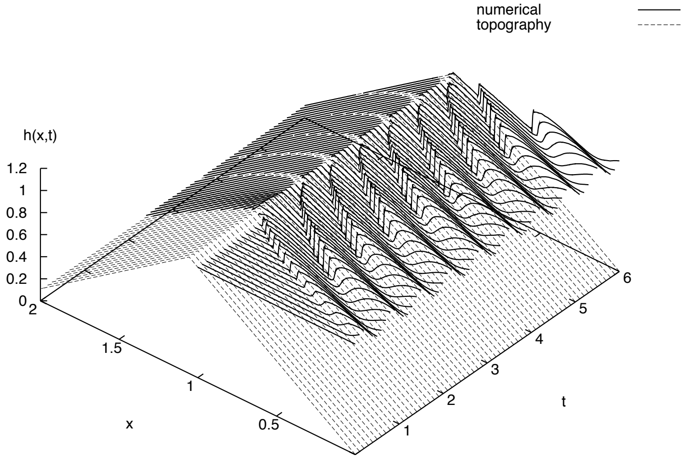
Discontinuous Galerkin Finite Element Method (Bokhove 2005)

- Flooding & drying: waterline movement $h(x,t) \to 0$.
- Allow only non-negative water depth $h \geq 0$ & changes in topology (sand banks).
- *Geometrical solutions merging & splitting*.



DGFEM SWE: Flooding & Drying 1D

Overtopping. Note regularity of solution!



DGFEM SWE: Flooding & Drying 2D

- 2D numerics by Ambati.
- Why so much accuracy needed on the beach?

Obliquely incident waves

DGFEM SWE: Sloshing Tank

Space-time Discontinuous Galerkin FEM

- Sloshing in tank with wavemaker.

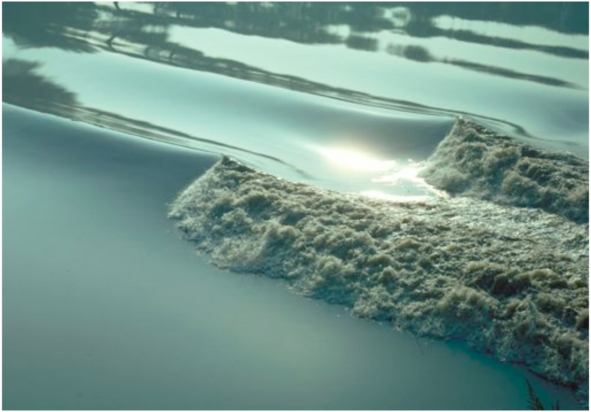
Movie loading please wait

4.3 New Hamiltonian Wave-Current Model

- Luke's VP for potential flow $(u, v, w)^T = \nabla\phi(x, y, z, t)$.
- New variational & Hamiltonian model with $(u, v, w)^T = \nabla\phi(x, y, z, t) + \mathrm{v}(x, y, t)$ & $\mathrm{v} = (\mathrm{v}_1, \mathrm{v}_2)^T$.
- Has pure potential & pure shallow water flow as limits (Cotter & B. 2010, $\sim$ Klopman 2010).
- Analytically/numerically: does it have breaking waves, balancing nonlinearity and dispersion?

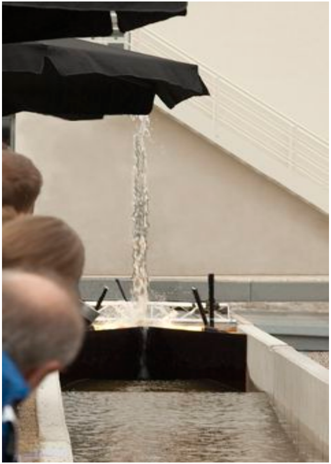
. . . with Breaking Waves?

- Breaking waves, dispersion & vorticity in new model?



4.4 Regular Exteme Wave: Bore Soliton Splash

My *BSS*: started as demonstration (ratio 10): relevant to Tohoku tsunami, rogue waves (ratio>2) & as test case.



5. Multi Scale Modeling Hele-Shaw Dynamics

Multiple scales: wave/particles $\sim$ 1Hz vs. beach 5min-1hr, wave length 0.4m vs. BL 0.001m:

- DNS: SPH/DGFEM Navier-Stokes solver plus Discrete Particle Models (in-house). Too costly.
- Classical coastal engineering models: wave-averaged or wave-resolving. Too phenomenological: particle entrainment/deposition laws, averaging linear in surf zone.
- Mixture theory for air/water/particles. No details on splashing and particles. *Under development*.
- Dedicated models for Hele-Shaw beach: to make contact with experiment. Too specific.

Wave-Averaged Coastal Models

Classical wave-averaged coastal engineering models:

- E.g., Garnier et al. (2006, 2008, 2010), McGall et al. (2009), Roelvink et al. (2010), Grimshaw & Osaisai 2011.
- All seem to do reasonably well, but sometimes sediment/sand conservation not closed. Wave-averaging in surf zone tends to be partially linear.
- Validate against Hele-Shaw beach dynamics.
- Use Hele-Shaw beach to get back to some fundamentals.

Wave-Resolving Coastal Models

Use Boussinesq-type wave models based on potential flow to resolve dispersive waves:

- Wave models, no sediment: Klopman et al. (2010), Cotter and B. (2010), Dutykh et al. (2011).
- Couple to boundary layer around particles.
- Add bores or wave breaking via mixture theory.

Mixture Theory

By analogy with granular systems:

- Use mixture theory to average fine-scale water-air interactions. Version derived based on potential theory and variational principles.
- Use mixture theory to average water-particle interactions, including porous flow.
- Use three-phase wave-resolving mixture theory for air, water, and particles.
- Advantage: feasibility in 3D. Requires multi-scale modeling & measurements to determine the closures.

Dedicated Theory for Hele-Shaw beach Dynamics

Employ quasi-2D nature:

- 2D hydrodynamics with linear momentum damping as diffusion.
- Porous flow in fixed, particle matrix.
- Boundary layer model around particles; measurements (Lee, Ramos, Swinney 2007).
- Depth- and wave averaged versions.
- Limited use as benchmark.

6. Conclusions

- Designed Hele-Shaw beach dynamics based on asymptotic analysis and 1D numerics.
- It works and is unique: breaking waves exist and can cause beach and dune formation.
- It is innovative. Quasi-2D modeling environment will stimulate mathematical advances from DNS to coarse-scale modeling, and test existing coastal models.

References

- My *www*: presentation, movies & eprints.
- B., Van der Horn, Van der Meer, Zweers & Thornton 2012: Breaking waves on a dynamic Hele-Shaw Beach. Paper for Shallow Flows Symposium, 10 pp.
- B., Gagarina, Zweers, Thornton 2011: Bore-Soliton-Splash: van Spektakel to Oceaangolf. Nederlands Tijdschrift v. Natuurkunde*url*.
- B., Zwart & Havemans 2010: *Fluid Fascinations*.