

On the beach

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# Mathematics of 'Hele-Shaw' Beach Evolution by Breaking Waves

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"Mathematics of Computational Science", University of Twente



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# 1. Introduction: Cutting Edge

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- Mathematical modeling of beach evolution by breaking waves will never be anything less than complicated: large-scale numerical computations are necessary, but because of the large number of (sand) grains involved, as well as the fine-scale water motion, brute-force computations will remain too demanding.
- Creative approaches to computational multi-scale modeling are required in order to reduce the degrees of freedom involved.

# Relevance

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- Beaches abound along world's coastlines.
- Understanding and prediction of beach dynamics lead to insights in order to forecast and prevent the dangers of flooding and erosion.
- Great advances made, e.g., Soulsby (1997), Calantoni et al. (2004, 2006), Garnier et al. (2006, 2008, 2010).
- But the laws of sand & sediment transport under breaking waves are still poorly understood.

# The Question

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## Is it possible to create a manageable mathematical modeling environment of beach dynamics?

- Equations water, air, particle motion in principle available, but DNS too costly.
- The desired modeling environment should permit research of a hierarchy of models.
- Including the range from DNS to depth-averaged models.
- Would such an environment also permit a **suitable laboratory experiment for validation?**

# A Cutting Edge Answer

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Yes<sup>2</sup>:

- Imagine we take a giant's knife, make two cuts to isolate a slice of beach . . . including sand and water, particle and wave motion.
- Place it between laterally periodic boundaries, or between two glass plates.
- Shrink the latter to table-top size: this Hele-Shaw beach experiment was demonstrated at [Fluid Fascinations](#).
- Qua Art & Qua Science show 2010: a tribute to the late Howell Peregrine.

# Fluid Fascinations

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Photo Severn Bore by DHP; painting by VZ:



## 2. Mathematical Design of Hele-Shaw Beach

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Advantages and **disadvantages**:

- Set-up unique as it focuses on particle motion by very nonlinear breaking waves.
- Immense reduction of dof's: quasi-2D.
- Innovative: allowing great visualisation & determination fundamental interactions, permitting research on a hierarchy of feasible mathematical modeling.
- **Is damping too severe due to the proximity of the glass plates?**
- **Determine minimal gap distance for which the dynamics is inertial & damping a secondary effect.**

# Asymptotic Analysis: width averaging

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Determine minimum gap width for which a broken wave can travel from wave maker to beach?

- 1. Focus on water motion.

Dynamics of 3D Navier-Stokes equations, scaled anisotropically with  $l$  &  $v$  scales  $(L, l, D)$  and  $(U, V, W)$ , pressure scale  $P_0 = \rho_0 U^2 / (Re \epsilon^2)$  with Reynolds number  $Re = UL/\nu$ , Froude number  $1/Fr^2 = gD/U^2$  and aspect ratios  $\epsilon = l/L \ll 1$  and  $\delta = D/L$ :

$$\begin{aligned} \partial_t w + u \partial_x w + v \partial_y w + w \partial_z w = & -\frac{1}{Re \epsilon^2 \delta^2} \partial_z p \\ & - \frac{1}{Fr^2 \delta^2} + \frac{1}{Re} (\partial_x^2 + \frac{1}{\epsilon^2} \partial_y^2 + \frac{1}{\delta^2} \partial_z^2) w \end{aligned} \quad (1a)$$

$$\partial_x u + \partial_y v + \partial_z w = 0, \quad (1b)$$

- Simplify:

$$\partial_t w + u \partial_x w + v \partial_y w + w \partial_z w = -\frac{1}{Re \epsilon^2 \delta^2} \partial_z p - \frac{1}{Fr^2 \delta^2} + \frac{1}{Re \epsilon^2} \partial_y^2 w \quad (2a)$$

$$\partial_x u + \partial_y v + \partial_z w = 0. \quad (2b)$$

- Tempting to assume balance between pressure gradient and dominant viscous terms, dimensionally:

$$u^* = -\frac{1}{2\nu} (\partial_{x^*} p^* / \rho_0) (l^2 - y_*^2) \quad (3)$$

$$w^* = -\frac{1}{2\nu} (\partial_{z^*} p^* / \rho_0 + g) (l^2 - y_*^2). \quad (4)$$

- 2. Consider ratio inertia terms/pressure gradient:

$$\frac{3\rho_0\tilde{u}^2}{L|\nabla_{xz}p|} = \frac{l^4|\nabla_{xz}p|}{3\rho_0\nu^2L} = \frac{l^4g\Delta h}{3\nu^2L^2} \quad (5a)$$

$$= \frac{l^4 \times 10 \times (4 \times 10^{-2})}{3 \times 10^{-12} \times (0.5)^2} \sim 0.1 \text{ to } 10 \quad (5b)$$

for  $l = 0.75$  to  $2$  mm. Hence, flow is **inertial**: there is **no global Hele-Shaw flow**.

- Pohlhausen (Rosenhead 1963) suggested Ansatz:

$$u = \frac{3}{2}\bar{u}\frac{(l^2 - y^2)}{l^2} \quad \text{and} \quad w = \frac{3}{2}\bar{w}\frac{(l^2 - y^2)}{l^2} \quad (6)$$

- ... a good approximation, cf. Wilson and Duffy (1998).

- 3. Average across gap: obtain 2D Navier-Stokes equations

$$\partial_t \bar{u} + \gamma \bar{u} \partial_x \bar{u} + \gamma \bar{w} \partial_z \bar{u} = -\frac{1}{\rho_0} \partial_x P - \frac{3\nu \bar{u}}{l^2} \quad (7a)$$

$$\partial_t \bar{w} + \gamma \bar{u} \partial_x \bar{w} + \gamma \bar{w} \partial_z \bar{w} = -\frac{1}{\rho_0} \partial_z P - g - \frac{3\nu \bar{w}}{l^2} \quad (7b)$$

$$\partial_x \bar{u} + \partial_z \bar{w} = 0, \quad (7c)$$

with  $y$ -independent pressure  $P = P(x, z, t)$ , width average  $\bar{u} = \int_{-l}^l u(x, y, z, t) dy / (2l)$  and  $\gamma = 6/5$ .

# Asymptotic Analysis: depth averaging

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4. Classical **depth-averaging** between bottom at  $z = b(x, t)$  and free surface at  $z = b(x, t) + h(x, t)$ .

- Kinematic free surface and bottom boundary conditions:

$$\partial_t h + \bar{u} \partial_x (b + h) - \bar{w} = 0 \text{ at } z = h(x, t) + b(x, t) \quad (8)$$

$$\partial_t b + \bar{u} \partial_x b - \bar{w} = 0 \text{ at } z = b(x, t). \quad (9)$$

- Use hydrostatic balance:

$$\partial_z p / (Re \epsilon^2) + 1 / Fr^2 = 0 \quad \text{or} \quad \partial_{z^*} P^* / \rho_0 + g = 0. \quad (10)$$

# Asymptotic Analysis: depth averaging

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- Average in depth and ignore (Reynolds) stress terms, gives **shallow water equations** with linear damping:

$$\partial_t(h\bar{u}) + \partial_x(\gamma h\bar{u}^2 + gh^2/2) = -gh\partial_x b - 3\nu h\bar{u}/l^2 \quad (11a)$$

$$\partial_t h + \partial_x(h\bar{u}) = 0 \quad (11b)$$

with  $h\bar{u}(x, t) = \int_b^{h+b} \bar{u}(x, z, t) dz$ .

- Represent breaking wave as shallow-water bore:  
discontinuity at  $x = x_b(t)$  has speed  $S = dx_b/dt$  satisfying

$$[h(u - S)] = 0 \quad [h(S - u)^2 + gh^2/2] = 0. \quad (12)$$

- I performed 1D numerical tests with a fixed beach, and a moving part of bottom as wave maker.

# ... depth averaging

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- Hence, the original design question: **Determine the minimum gap width for which a broken wave can travel to from wave maker to beach?** leads to the simpler question:
- **For which gap width  $2l$  can a bore generated by a wave maker reach the end of the beach?**
- Answer (...): for a beach of length  $\sim 0.5\text{m}$ , gap width  $2l > 1.5\text{mm}$ .
- Given the availability of zeolite particles with  $d = 1.80 \pm 0.05\text{mm}$ , we chose  $2l = 2\text{mm}$ .
- We then simply build two experimental set-ups (Zweers, B., Thornton), to test my asymptotic calculations.
- Rest, whether particles could move, is history.

### 3. Beach and Dune Formation: in the Lab

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Wave-sand dynamics in a vertical “plane”:

- 2 vertical glass plates:  $0.6 \times 0.3 \times 0.002$  or  $1 \times 0.3 \times 0.002\text{m}^3$
- filled with water & heavier zeolite particles  $d = 1.8\text{mm}$
- wavemaker: moving welding rod 1.5mm thin;  $\sim 1\text{Hz}$
- “2D” dynamics in Hele-Shaw cell.



# Sketch Hele-Shaw beach

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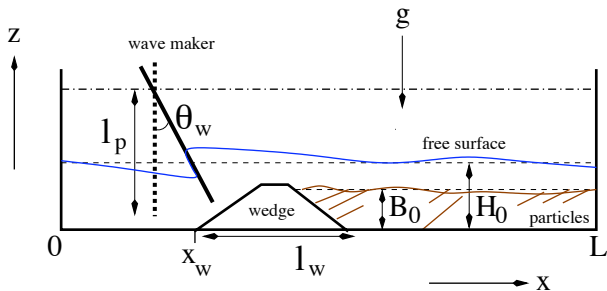
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Sketch of Hele-Shaw cell with wedge, waterline, particle bottom, and wave-maker rod in two positions.

# Wave-Maker

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## *Wave-maker*



# Wave Types

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Breaker *wave types* (Peregrine 1983 ARFM) observed:

- **Spilling**: white water at wave crest spills down front face sometimes with projection of small jet
- **Plunging**: wave's front face overturns, prominent jet at base wave, causing large splash
- **Collapsing**: lower portion front face overturns, behaves like truncated plunging breaker
- **Surging**: significant disturbance smooth profile occurs only near moving shoreline
- ... **Shore break**: whole face from trough to crest vertical with little/no water in front.

63, 115, 133, 8s

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# Beach Formation

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- Monochromatic wave frequency 0.6m–cell: *beach creation*.
- Monochromatic 0.6hz wave frequency in 1m–cell; 30s stills: *formation of sand waves & beach*.



# Dune Formation

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- Alternating between wave frequencies 0.6 and 0.9Hz: *dune creation*, ill-understood, hysteretic effects.



# Modeling challenges

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Given these design calculations and experimental results, the goal becomes:

- to predict the dynamics in the Hele-Shaw cell,
- with models that can be extended to yield feasible 3D-predictions.

Consider the degrees of freedom:

- DNS simulations:  $\geq 10^3$  per particle; tank  $250 \times 40 \times 1 d^3 = 10^4 d^3$ .
- Leading to  $10^7 - 10^8$  dof's for Navier-Stokes and DPMs! Perhaps feasible in quasi-2D but not in 3D.
- Again, ... a hierarchy of reduced models is required ....

## 4. Numerical Modeling Expertise

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Whitham (Lin. & Nonlin. Waves 1974) gives **2 wave types**:

- **Hyperbolic waves**:

$$\partial_{tt}\phi - \partial_x(c^2(x)\partial_x\phi) = 0$$

with  $\phi = \phi(x, t)$ , wave speed  $c(x) = \pm\sqrt{gH(x)}$ , still  
depth  $H = H(x)$  & gravity cst  $g$ .

- **Dispersive waves**: system with linear dispersive solutions

$$\varphi(x, t) = A \cos(kx - \omega t).$$

# Linear hyperbolic waves

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**Variational principle** with velocity  $u = \partial_x \phi(x, t)$ , free surface deviation  $\eta = \eta(x, t)$ :

$$0 = \delta \int_0^T \mathcal{L}[\phi, \eta] dt = \delta \int_0^T \int_0^L \eta \partial_t \phi + \frac{1}{2} H(x) (\partial_x \phi)^2 + \frac{1}{2} g \eta^2 dx dt.$$

Require functional derivatives defined by:

$$\begin{aligned} \delta \int_0^T \mathcal{L}[\phi, \eta] dt &= \lim_{\epsilon \rightarrow 0} \int_0^T \frac{\mathcal{L}[\phi + \epsilon \delta \phi, \eta + \epsilon \delta \eta] - \mathcal{L}[\phi, \eta]}{\epsilon} dt \\ &= \int_0^T \int_0^L (\partial_t \phi + g \eta) \delta \eta - (\partial_t \eta + \partial_x (H(x) \partial_x \phi)) \delta \phi dx dt. \end{aligned}$$

w.  $u(0, t) = u(L, t) = 0$  or flow on 1-torus. **Arbitrariness** . . . :

$$\delta \eta : \partial_t \phi + \frac{\delta \mathcal{H}}{\delta \eta} = 0 \quad \text{and} \quad \partial_t \eta - \frac{\delta \mathcal{H}}{\delta \phi} = 0. \quad (13)$$

# Linear hyperbolic waves

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## Linear shallow water waves:

$$\begin{aligned}\partial_t \phi + g\eta &= 0 \quad \text{and} \quad \partial_t \eta + \partial_x (H(x) \partial_x \phi) = 0 \\ \implies \partial_{tt} \phi - \partial_x (c^2(x) \partial_x \phi) &= 0.\end{aligned}$$

If  $H(x) = H_0$  cst then  $c_0 = \pm \sqrt{gH_0}$ ; 1D wave eqn:

$$\partial_{tt} \phi - c_0^2 \partial_{xx} \phi = 0$$

$\rightarrow \phi = \phi_R(x - c_0 t) + \phi_L(x + c_0 t)$ ;  $\phi_R$  satisfies advection eqn

$$\partial_t \phi + c_0 \partial_x \phi = 0. \tag{14}$$

# Beauty

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**Beauty** of variational formulation:

- it encompasses system succinctly into one integral
- it is starting point for compatible numerical discretization
- energy conservation and “phase space conservation”.

**Energy conservation**, by skew-symmetry:

$$\frac{d\mathcal{H}}{dt} = \int_0^L \frac{\delta\mathcal{H}}{\delta\eta} \partial_t \eta + \frac{\delta\mathcal{H}}{\delta\phi} \partial_t \phi \, dx = 0$$

with Hamiltonian/energy:

$$\begin{aligned} \mathcal{H} &= \int_0^L \frac{1}{2} H_0 (\partial_x \phi)^2 + \frac{1}{2} g \eta^2 \, dx \\ &\approx \Delta x \sum_{i=1}^N \frac{1}{2} H_0 \frac{(\phi_i - \phi_{i-1})^2}{\Delta x^2} + \frac{1}{2} g \eta_i^2 = H_d \Delta x. \end{aligned}$$

**Phase space conservation:** invariance phase volume (Liouville).

- Consider discretized system ( $i = 1, \dots, N$ ):

$$\delta\phi_i : \dot{\eta}_i = \frac{\partial H_d}{\partial \phi_i} = -H_0 \frac{\phi_{i+1} - 2\phi_i + \phi_{i-1}}{\Delta x^2}$$

$$\delta\eta_i : \dot{\phi}_i = -\frac{\partial H_d}{\partial \eta_i} = -g\eta_i$$

$$H_d = \sum_{i=1}^N \frac{1}{2} H_0 \frac{(\phi_i - \phi_{i-1})^2}{\Delta x} + \frac{1}{2} g \eta_i^2. \quad (15)$$

- $2N$ -dimensional phase volume  $\Omega = \int \dots d\phi_i d\eta_i \dots$  conserved.

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Consider linear water waves in vertical plane using **linearized Luke's variational principle**.

- Velocity  $(u, w)^T = \nabla_{xz}\phi = (\partial_x\phi, \partial_z\phi)^T$ ; potential  $\phi = \phi(x, z, t)$ .
- Rectangular  $D$ ; free surface  $z = H_0 + \eta(x, t)$  with  $\phi_s = \phi(x, H_0, t)$ ;  $\hat{\mathbf{n}} \cdot \nabla_{xz}\phi = 0$  at walls ( $g = H_0 = 1$ ):

$$\begin{aligned} 0 &= \delta \int_0^T \mathcal{L}[\phi, \phi_s, \eta] dt \\ &= \delta \int_0^T \int_0^L \eta \partial_t \phi_s + \frac{1}{2} \eta^2 + \int_0^{H_0} \frac{1}{2} |\nabla_{xz}\phi|^2 dz dx dt \\ &= \int \int (\partial_t \phi_s + \eta) \delta \eta - (\partial_t \eta - (\partial_z \phi)_s) \delta \phi_s \\ &\quad - \int \delta \phi \nabla^2 \phi dz dx dt. \end{aligned}$$

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- Subs. Ansatz  $\eta \propto e^{i(kx-\omega t)}$  &  $\phi \propto \hat{\phi}(z)e^{i(kx-\omega t)}$  into eqns

$$\nabla^2 \phi = 0 \text{ and } \partial_t \eta - (\partial_z \phi)_s = 0, \partial_t \phi_s + g\eta = 0 \text{ at } z = H_0$$

- to obtain ... **classical water wave dispersion relation**

$$\omega = \omega(k) = \pm(gk \tanh(kH_0))^{1/2}. \quad (16)$$

- Waves with different wave numbers  $k$  travel with different wave speeds  $c_w = \omega/k$ .
- For shallow water waves  $kH_0 \rightarrow 0$  so  $c_w = \pm\sqrt{gH_0} = cst$ .
- .....

# Hyperbolic shallow water waves

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## Nonlinear shallow water equations in 1D:

- Bottom  $b = b(x)$ , velocity  $u = u(x, t)$ , depth  $h = h(x, t)$

$$\partial_t h + \partial_x(hu) = 0$$

$$\partial_t(hu) + \partial_x(hu^2 + gh^2/2) = -gh\partial_x b \quad (17)$$

- or

$$\partial_t h + \partial_x(hu) = 0, \quad \partial_t u + u\partial_x u = -g\partial_x(h + b)$$

- or, ( $b = x$ ) **Riemann invariants**  $\alpha = u + 2a + gt$  and  $\beta = u - 2a + gt$

$$\partial_t(u \pm 2a + gt) + (u \pm a)\partial_x(u \pm 2a + gt) = 0$$

- Note that  $a = \sqrt{gh}$  has linear counterpart  $c_0 = \sqrt{gH_0}$ .
- Interpretation:  $\alpha, \beta$  conserved on their characteristics:  
 $d(u \pm 2a + gt)/dt = 0$  on  $dx/dt = u \pm a$ .

# Overtaking

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But SWE without more rules **admit multivalued solutions**, e.g.:

- Take  $b = 0$  and  $\alpha = u + 2a = K$  constant then  $u = K - 2a = K - 2\sqrt{gh}$ .
- Reduction to one (the other) Riemann invariant with characteristic  $q(x, t) = u - a = K - 3a$ , and **Burgers' eqn**:

$$\partial_t(K - 4a) + (K - 3a)\partial_x(K - 4a) = 0 \implies \partial_t q + q\partial_x q = 0.$$

- Overtaking solutions since  $q(x, t) = q_0(x - q(x, t)t)$ , or

$$\frac{dq}{dt} = 0 \text{ on } \frac{dx}{dt} = q.$$

# Overtuning

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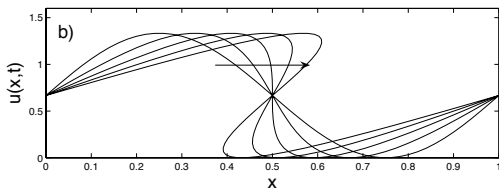
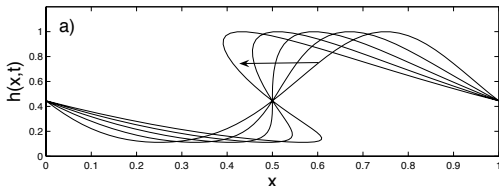
Conclusions

Example.

- Transform back to original variables:

$$h(x, t) = (K - q(x, t))^2 / 9 \quad \text{and} \quad u(x, t) = (K + 2q(x, t)) / 3.$$

- $t = (0, 1, 2, 3, 4) / (4\pi)$  for  $q_0(x) = \sin(2\pi x)$  &  $K = 2$ .



# Shocks or Bores

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Conclusion: **augment SWE with weak solution for shocks:**

- Consider shock at  $x = x_b(t)$  with speed  $S = dx_b/dt$ .
- **Dissipate energy:** integrate (17) around & limit to  $x_b(t)$ :

$$\begin{cases} \lim_{\epsilon \rightarrow 0} \int_{x_b - \epsilon}^{x_b + \epsilon} \partial_t h + \partial_x(hu) dx = 0 \\ \lim_{\epsilon \rightarrow 0} \int_{x_b - \epsilon}^{x_b + \epsilon} \partial_t(hu) + \partial_x(hu^2 + gh^2/2) - gh\partial_x b dx = 0 \\ -S[h] + [hu] = 0, \quad -S[hu] + [hu^2 + gh^2/2] = 0 \end{cases}$$

with  $[\cdot]$  (free surface) jump across shock.

- Def: moving shock in hydraulics = **“bore”**; steady one = **“hydraulic jump”**.

# Bore

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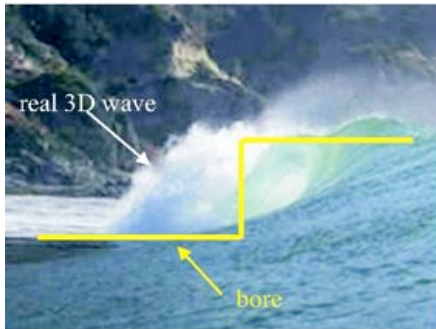
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A bore is a **mathematical abstraction** of a complicated 3D breaking wave



# Discontinuous Galerkin FE Shallow Water Flows

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- 1D & 2D numerics with flooding & drying (Ph.D.'s Tassi, Ambati, Gagarina).
- Why so much **accuracy** needed on the beach?

Obliquely incident waves

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## Space discontinuous Galerkin FEM

- Creation of vorticity  $\partial_x v - \partial_y u$  by  
*bore over conical hill:*

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## Space discontinuous Galerkin FEM

- Creation of vorticity  $\partial_x v - \partial_y u$  by  
*bore over conical hill:*

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## Space-time Discontinuous Galerkin FEM

- Sloshing in tank with wavemaker

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## Space-time Discontinuous Galerkin FEM

- Sloshing in tank with wavemaker

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# Dispersive breaking waves?

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Conclusions

- **Luke's VP** for potential flow  $(u, v, w)^T = \nabla\phi(x, y, z, t)$ .
- New intermediate VP with  $(u, v, w)^T = \nabla\phi(x, y, z, t) + v(x, y, t)$  &  $v = (v_1, v_2)^T$ .
- New VP: pure potential & pure shallow water flow as limits (**2 for 1**) (Cotter & B. 2010, ~ Klopman 2010).
- Reformulated as Hamiltonian system with Ph.D. Elena Gagarina.
- New VP: breaking waves, balancing nonlinearity and dispersion?
- New VP: numerical counterpart w. **wave breaking?**

# Compatible numerical schemes

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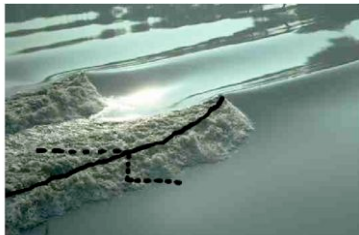
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Discrete (nonlinear) Luke's variational principle.

- **Discretization immediate:**  
substitute (DG)FEM expansions directly in VP:  
Hamiltonian PDE's  $\rightarrow$  Hamiltonian ODE's.
- But, *mesh movement* included in VP.
- *Wave maker movement* included in VP.
- Proof of convergence in progress: Iszak & JvdV.
- Inclusion breaking waves and **vorticity** into new model:



## 5. Modeling Hierarchy?

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Scales: wave/particles  $\sim$  1Hz, beach 5min-1 hour:

- DNS: SPH/DGFEM Navier-Stokes solver plus Discrete Particle Models (in-house).
- Two-phase mixture theory for air and water, reducing to potential flow for air/water without overturning.
- Three-phase **wave-resolving mixture theory** for air, water, and particles.
- Porous flow in particle matrix, 2D water flow away from & BL-approach near particles (Lee, Ramos, Swinney 2011).
- Classical **wave-averaged** civil-engineering models for comparison, e.g. Grimshaw & Osaisai 2011.

## 6. Conclusions

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- Designed **Hele-Shaw beach dynamics** based on asymptotic analysis and 1D numerics.
- **It works and is unique**: breaking waves exist and can cause beach and dune formation.
- **It is innovative**. Quasi-2D modeling environment will stimulate mathematical advances from DNS to coarse-scale modeling.

# Fluid Fascinations 2010

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# Bore Soliton Splash

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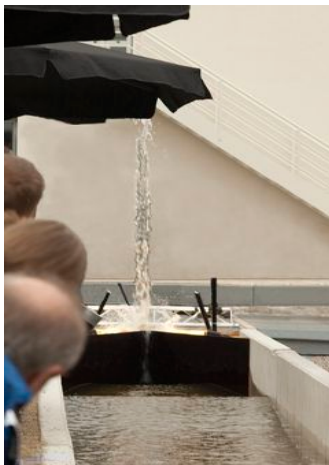
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**BSS**: started as "a bit of fun" but is relevant to Tohoku tsunami, rogue waves, & as test case (NTvN Dec 2011).



# References

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- Presentation and movies online on my web page.
- Beach stuff: *Geophys. Res. Lett.* in preparation.
- B., Gagarina, Zweers, Thornton 2011:  
Bore-Soliton-Splash: van Spektakel to Oceaangolf.  
*Nederlands Tijdschrift v. Natuurkunde*, Dec.
- Cotter & B. 2010: Variational water wave model with  
accurate dispersion and vorticity. *J. Eng. Mech.* **67**.
- B., Zwart & Havemans 2010: Fluid Fascinations.

# Stormy shallow water bores

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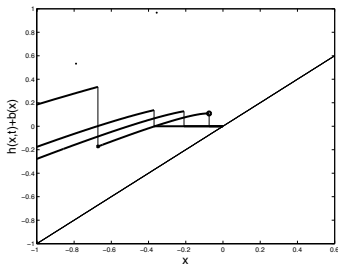
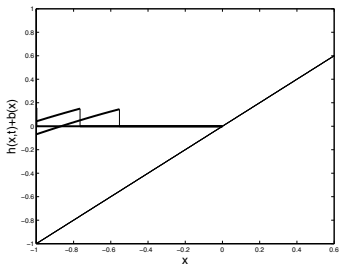
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Conclusions

- Semi-analytical bore solutions w. asymptotic extensions (2D): **longshore currents generated by storms** (obliquely incident breaking waves) (Antuono 2010):



# Stormy shallow water bores/challenge

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## ■ Stormy bores on a beach:

