Matcont Tutorial: ODE GUI version

Hil Meijer

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"If you want to get credit for solving a complicated mathematical problem, you will have to provide a full proof. But if you're trying to make something as easy as possible, you want to make it foolproof–so simple even a fool could couldn't screw it up."[1]

1 Introduction and Installation

This tutorial tries to explain the basics of how to use the numerical bifurcation package MATCONT by going through an example. This is a condensed and short overview, i.e. not all details and philosphy are explained, and this tutorial cannot substitute a complete course. We aim at explaining how you can use the software so we assume a basic knowledge of bifurcation theory. As with many practical things one learns, you will only master it by performing the procedures yourself. So hands on!

If you want to have more precise details, then go to the matcont website on sourceforge. Here you find more elaborated tutorial sessions. You can also look at course material of "Introduction to Numerical Bifurcation Analysis" on the website of Yuri Kuznetsov, under teaching. Another reference is [2].

We try to cover the following procedures in this tutorial

- Installation, including steps on installing a compiler
- System definition and simulation
- Continuation of equilibria in one parameter
- Continuation of codim 1 bifurcations of equilibria in two parameters
- Starting Limit Cycle from a Hopf point or time simulation
- Continuation of codim 1 bifurcations of Limit Cycles
- Computing homoclinic orbits

[1] Every one computer is different from the other. So, although these instructions have been checked step by step, do not be surprised if your computations take a different direction. Just try again but now in the other direction.

[2] New features of the software MatCont for bifurcation analysis of dynamical systems. A. Dhooge, and W. Govaerts, Yu.A. Kuznetsov, H.G.E. Meijer and B. Sautois, Mathematical and Computer Modelling of Dynamical Systems, Vol. 14, No. 2, pp 147-175, 2008

1.1 Starting Matcont

1. Please let Matlab be installed, and download matcont (latest version http://sourceforge.net/projects/matcont/files/matcont/matcont6p3/). This is the GUI-version, and should not be confused with the version for maps or command line. Unzip the pacakge to a folder of your choice.

2. For periodic orbits we use compiled C-code to speedup the computation of the Jacobians. Over the years, the support of MATLAB for compilers has reduced now relying on manual installation of a compiler by the user. To overcome this we have precompiled the necessary files. You can download these from http://sourceforge.net/projects/matcont/files/Auxiliaries/MEXfiles/

2a. Choose your platform mexw32, mexw64, mexmaci64. You can check this with “mexext” on the Matlab command line if you are not sure. Download the 7 files and put them in the folder LimitCycle of your matcont folder.

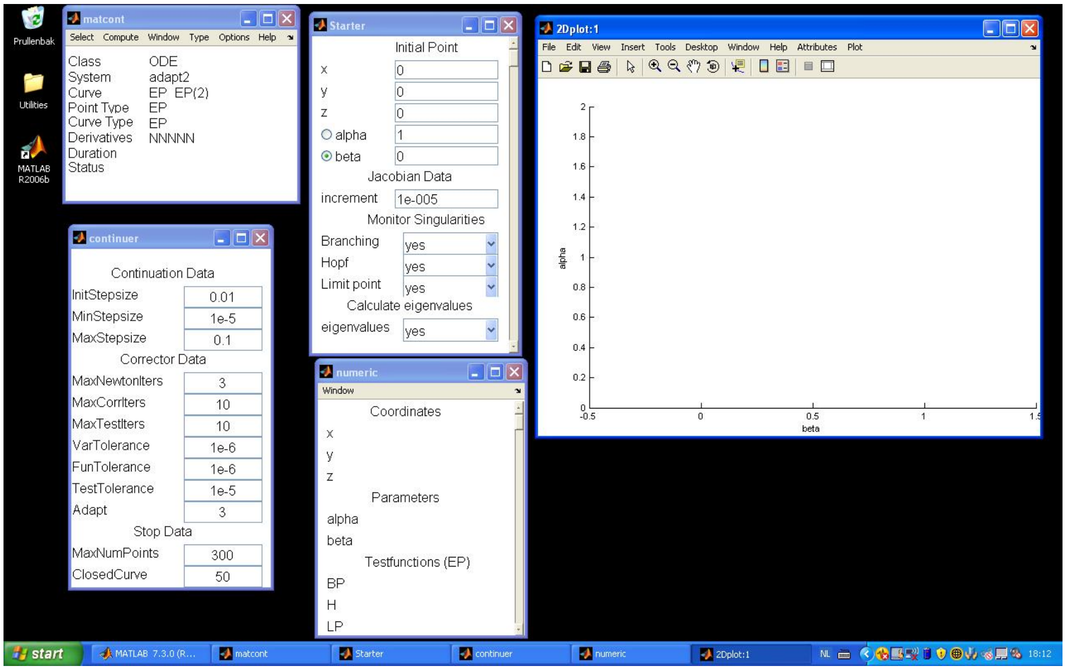
2b. If for some reason, option 2a does not work you should have a compiler [3]. If you receive complaints about a compiler, type ‘mex -setup’ on the command line and choose the compiler you installed or lcc, the Matlab compiler. Start matcont again.

3. $\Longrightarrow$ Switch to the matcont directory and type ‘matcont’. Some windows should appear as in Figure 1. You can now start using Matcont.

Now several windows will open with a standard system called adapt2. A typical screenshot is shown in Figure 1. One window is titled matcont and has several menu options. For instance, to end your matcont session, choose ‘Select’ in the matcont window and then ‘Exit’. Hereafter we will indicate this with *matcont*:**Select**|**Exit**. Alternatively one clicks on the close button of the matcont window. You matlab working folder must be the matcont folder, otherwise not all windows will close. If you actually ended your session, type ‘matcont’ to restart it.

[3] We will try to implement vectorization in the near future. For now the situation is as follows

1. Windows XP (or Vista/7 with 32-bits processor), works fine, run mex -setup and choose the lcc compiler.
2. Windows 7: You will need to install Visual Studio 2010 Express and SDK 7, as indicated in http://www.mathworks.com/support/compilers/R2010b/win64.html (Windows 8& 10 not tested)
3. MAC: If at mex -setup you can select gcc without problems, proceed. Otherwise you have to install the XCode package.
4. Linux: this has its gcc compiler properly installed on most distributions.



matcont

Class	ODE
System	adapt2
Curve	EP EP(2)
Point Type	EP
Curve Type	EP
Derivatives	NNNNN
Duration	
Status	

Starter

Initial Point

x	0
y	0
z	0
○ alpha	1
● beta	0

Jacobian Data

increment	1e-005

Monitor Singularities

Branching	yes
Hopf	yes
Limit point	yes

Calculate eigenvalues

eigenvalues	yes

continuer

Continuation Data

InitStepsize	0.01
MinStepsize	1e-5
MaxStepsize	0.1

Corrector Data

MaxNewtonIters	3
MaxCorrIters	10
MaxTestIters	10
VarTolerance	1e-6
FunTolerance	1e-6
TestTolerance	1e-5
Adapt	3

Stop Data

MaxNumPoints	300
ClosedCurve	50

numeric

Coordinates

x
y
z

Parameters

alpha
beta

Testfunctions (EP)

BP
H
LP

Figure 1: A typical matcont startup screen

2 Equilibria in Lorenz84

In this tutorial we will first investigate some simple equilibrium bifurcations in the Lorenz-84 model, which can be seen as a metaphor of the large-scale atmospheric circulation, given by

$$\left\{\begin{array}{rl} x' = & -y^2 - z^2 - ax + aF, \\ y' = & xy - bxz - y + G, \\ z' = & bxy + xz - z. \end{array}\right. \qquad (1)$$

2.1 Specifying a new model

We need to tell Matcont what model we want to study. We will now go through this process. The result will be that a m-file is created and placed in the folder Systems of your Matcont folder. This file has the name of your system, and a call to this m-file allows Matcont to evaluate the right-hand side (RHS) of your ODE.

To specify the differential equations of the sytem click '(*matcont*):**Select**|**Systems**|**New**' and a window titled 'System' appears with fields 'Name system', 'Coordinates', 'Parameters', 'Time', options for derivatives and a large box. The exact appearance of the System window depends on the operating system and the availability of the Matlab Symbolic Toolbox. If the latter is present, the lowest line will read 'symbolically'. If possible, one should always choose to compute first, second and third order derivatives symbolically as this is the most reliable and fastest option [4]. If you have a large system (many variables) or complex functions, it may be enough to have just first order derivatives. Computing higher order derivatives symbolically may take too much time.

Fill the fields as follows

- Name: Lorenz84
- Coordinates: x,y,z
- Parameters: a,b,F,G
- Time: t (we do not use it here.)
- Select symbolic derivatives up to order 3 (if possible).
- In the big field specify the ODE:

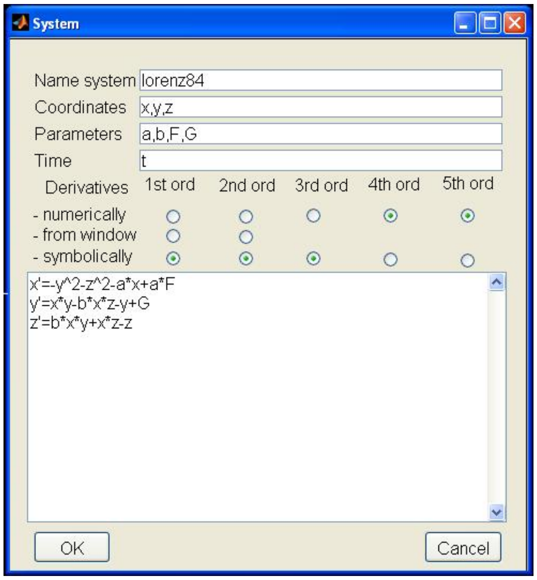
```
x'=-y*y-z*z-a*x+a*F
y'=x*y-b*x*z-y+G
z'=b*x*y+x*z-z
```

To avoid typos you may also use the input from the file www.math.utwente.nl/∼meijerhge/MT.txt (or google Hil Meijer and go to my software page). Now you can copy-paste all the required fields. You will have something like in Figure 2 and click **Ok**.
If you want to change the system or correct the input choose '(*matcont*):**Select|Systems|Edit/Load**', select the system you want to edit and press 'Edit'.
We have observed the following typical mistakes (in addition to simple typos):

- Some names (e.g. C) are internal matlab commands or variables and cannot be used.
- Do not use spaces between variables or parameters in their input fields
- Make sure that multiplication is written explicitly with * or you get errors like 'Unbalanced or misused parentheses or brackets' or 'error using ==> eval, Index exceeds matrix dimensions'.
- Specify the differential equations in the same order as the order of the coordinates.

If you filled the fields correctly, the System window now disappears, and you will see that the field '(*matcont*):System' displays Lorenz84 as the current system. If selected, the information field '(*matcont*):Derivatives' shows the string SSSNN meaning that symbolic 3rd order derivatives of the RHS will be used.

[4] The first and second are mostly used in the continuation, the third order is useful for computing normal forms accurately.



Name system lorenz84

Coordinates x,y,z

Parameters a,b,F,G

Time t

Derivatives	1st ord	2nd ord	3rd ord	4th ord	5th ord
- numerically	○	○	○	●	●
- from window	○	○			
- symbolically	●	●	●	○	○

```
x'=-y^2-z^2-a*x+a*F
y'=x*y-b*x*z-y+G
z'=b*x*y+x*z-z
```

Figure 2: Specifying a new model

2.2 Time Simulations

Click '(*matcont*):**Type**|**Initial point**|**Point**' to initialize the computation of an orbit starting from a point. In the matcont window the curve type is now P_O, every curve type has a similar meaning. Two extra windows '*Starter*' and '*Integrator*' appear. The first is to specify initial conditions and parameter values, the second to select the numerical integrator and change its settings. In the starter window set $a = .25, b = 4, F = .5, G = .5$. We also want to see how the orbits look like and for this we open a window '(*matcont*):**Window**|**Plot**|**2D-plot**'. We want to see the x, y-plane. In the window '**Layout**|**Variables on axes**', we select x as abscissa (standard) and y as ordinate, press 'OK'. Now we should adjust to the range of values that $x(t)$ and $y(t)$ will have. Using '(*2Dplot:1*):**Layout**|**Plotting region**', set the axes range from 0 to .5 for the abscissa and -.5 to .5 for the ordinate. In 'Integrator' change 'Interval' to 50 and 'MaxStepsize' to 0.1.

After these preparations we are ready to compute some orbits: Select '(*matcont*): **Compute**|**Forward**'. Starting from $(x, y, z) = (0, 0, 0)$ converges to $(-.13444588, .359872257, -.170597270)$. You can monitor the orbit and its final values by opening via numeric window. Select '(*matcont*): **Window**|**Numeric**' and recompute the orbit. You can try also other initial points and check that these also converge to this point. So we have a(n apparent) global attractor.

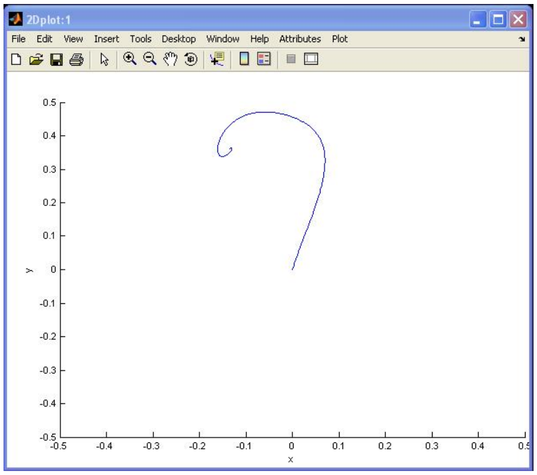


Figure 3: A spirally converging orbit

2.3 Continuation of Equilibria

We are now going to study the effect on the stability of the equilibrium when varying either F or G. Select the last point via '(*matcont*):Initial point' and then choose the last point on the previous curve. Change the curve type to EP_EP by selecting '(*matcont*):**Type**|**Initial point**|**Equilibrium**'. Activate the parameter F by clicking on the button next to it. In the 2Dplot:1 window change the abscissa to the parameter F via '(*2Dplot:1*):**Layout**|**Variables on axes**'. Also change the plotting region for F to 0 to 2 and for y to -1 to 1.

The stability changes upon the variation of F thus we would also like to see how the eigenvalues of the Jacobian at the equilibrium develop. For this we select '(*Numeric*):**Window**|**Layout**' and highlight the option 'eigenvalues'(it changes to capitals) and press 'OK'. Select '(*matcont*): **Compute**|**Forward**'.

NOTE: *In the lower left corner a small window appears with buttons 'Pause','Resume' and 'Stop'. Whenever a bifurcation is detected, MATCONT pauses and shows some relevant information. In the sequel a few such points will be detected, but we will not always indicate to resume the computation.*

At $F = 1.1345161$ the message 'Limit point' appears in the status field of the matcont window. Here one eigenvalue has zero real part(Check that in the numeric window!) and the curve(branch) of equilibria has a turning point, on one side the equilibria are stable on the other unstable. Press the 'Resume' button in the lower left of your screen or press space to continue the computation of the now unstable branch. At $F = 0.73193833$ another Limit point is found and the unstable branch turns into a stable one. When $F = 1.0757564$ is reached two complex conjugate eigenvalues have zero real part and a Hopf bifurcation occurs. The 2Dplot should now look like Figure 4.

If the computation goes to fast you can select to pause the computation of the

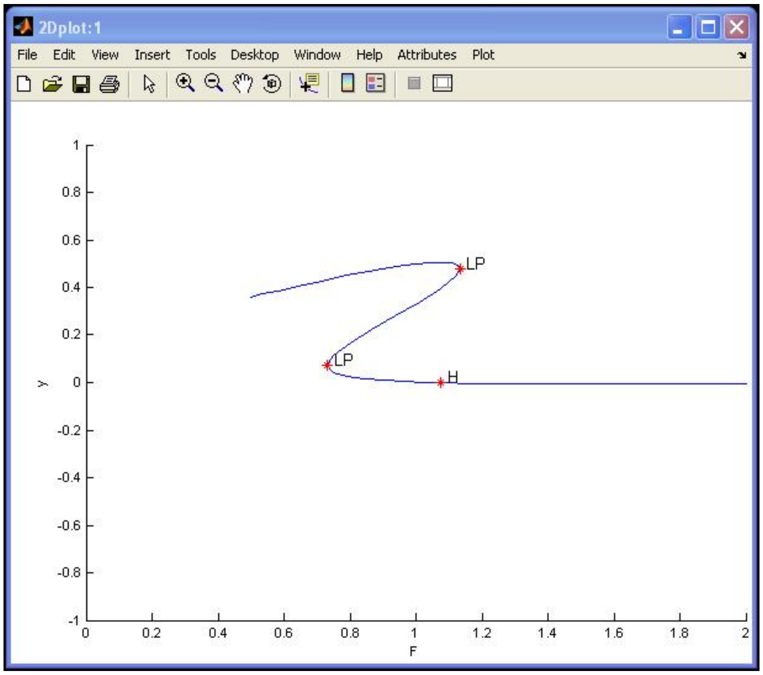


Figure 4: A branch of equilibria in the (F, y)-plane displaying bistability.

EP_EP curve after each computed point. For this choose '(*matcont*):**Options**|**Pause**|**At each point**'. (Do not forget to change it to 'At special points' after this computation.

Now we turn to the command line, i.e. the Matlab window, where you most probably now have the following information:

```
first point found
tangent vector to first point found
label = LP, x = ( 0.085554 0.479610 0.179485 1.134516 )
a=-9.177168e-001
label = LP, x = ( 0.461768 0.072707 0.249512 0.731938 )
a=-4.844850e-001
label = H , x = ( 1.015103 -0.000458 0.123138 1.075756 )
First Lyapunov coefficient = -9.496611e+000
elapsed time  = xxx secs
npoints curve = 300
```

The first two lines indicate that the continuation has started, and the last two that it has ended. Any other messages are displayed here, for instance the three bifurcations together with their normal form coefficients. This information is useful as it indicates the non-degeneracy of the bifurcations. First, the two

Limit Point bifurcations are shown, x indicates the three variables x, y, z and the value of the active parameter F of the bifurcation point. Then the normal form coefficients are given, note that for a Limit Point the sign of this coefficient is not unique [5]. Finally, we know that the limit cycle born from the Hopf bifurcation is stable as the first Lyapunov coeffiecient is negative.

2.4 Continuation of codim 1 bifurcations of equilibria

Figure 4 showed a phenomenon called hysteresis or bistability. This is an important feature of nonlinear (differential) equations. In parameter space regions of bistability are usually delimited by a wedge of Limit Point bifurcations which we will compute here.

Select one of the Limit points found in the equilibrium continuation as initial point '(*matcont*):**Select|Initial point**'. Activate F and G as active parameters. Close the 3D-plot and re-open the 2D-plot. Change the ordinate of the 2D-plot to the parameter G and its range to 0 to 2. Now select '(*matcont*):**Compute|Forward**' and when finished also 'Backward'. At $(F, G) = (0.4663649, 0.2919545)$ a Cusp bifurcation (CP) and at $(F, G) = (1.6840517, 1.6829686)$ a Zero-Hopf bifurcation is detected. You have now computed the wedge within which you have bistability.

Finally we also compute a curve of Hopf bifurcations: select a Hopf point analogously as for the Limit point as initial point. Change the curve type to H_H via '(*matcont*):**Type|Curve|Hopf**'. Here we set F and G as active parameters. Press '**Compute|Forward**' and '**Compute|Backward**' to obtain the diagram as in Figure 5.

2.5 Challenge problem:

Consider the following 2D prey-predator model from mathematical ecology

$$\left\{ \begin{array}{lcl} x' & = & rx(1-x) - fracxyx + a, \\ y' & = & -cy + \frac{xy}{x+a} - \frac{dy^2}{y^2+b^2}. \end{array} \right.$$

We set $a = 0.6, \quad b = c = 0.25$ and also $r = 2$ and $d = 0.1$.

Produce a bifurcation diagram with all Limit Point and Hopf bifurcation curves you can find in the (r, d)-parameter plane. That is, keep a, b, c fixed and use r and d as free parameters. You will have to specify a new system (rename c to cc), and to do all continuation of equilibria and LP and H points, just as we discussed. The nicest solution (on Tuesday morning) will get a small prize.

[5] Can you understand why from the formula $a = \langle p, B(q, q) \rangle$?

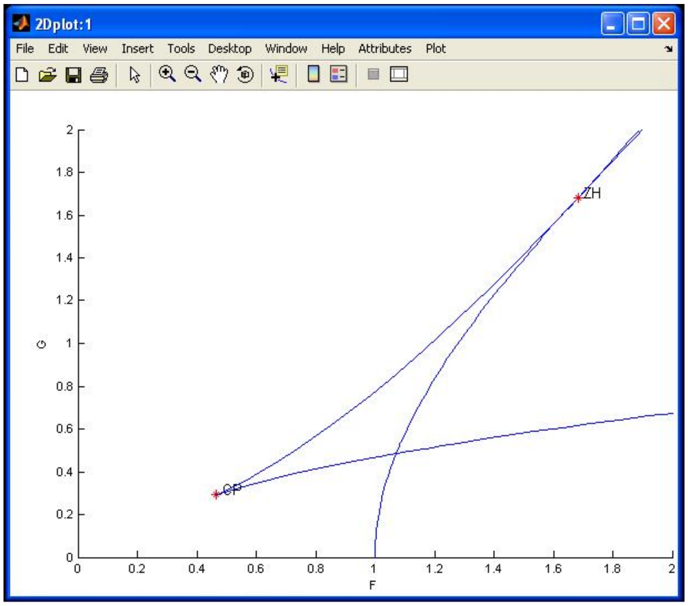

2Dplot:1
File Edit View Insert Tools Desktop Window Help Attributes Plot
ZH
CP
G
F
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2

Figure 5: Bifurcation curves in the (F, G)-plane with codimension 2 points.

3 Continuation of Limit Cycles

3.1 Starting from a Hopf point

From the continuation of equilibria we found two LP points and one Hopf point. We are now going to compute a branch of limit cycles starting from the Hopf point. We will do this in one parameter F. This means that we add the period as additional parameter. By default, and for good reasons, detection of bifurcations is switched off. Here we will encounter a PD bifurcation. So we will need to activate the monitoring of that testfunction. Also you may find that plotting the cycles takes too much time. If you want to speed it up, close the 2Dplot.

Execute:

- Select the Hopf point using ‘(*matcont*):**Select**|**Initial point**’ as new initial point.
- Activate the parameter F and the period by clicking on the buttons in the Starter window.
- In the starter window set Monitor Singularities for Period-doubling to **Yes**
- Press‘(*matcont*):**Compute**|**Forward**’.

- When a Period-Doubling bifurcation is detected at $F = 7.129405$ and $F = 12.8991$, inspect the normal form coefficient b of the Period-Doubling bifurcation on the command line.
- press 'Resume' and wait until it has finished.

Results To inspect the results we will use a 3D plot in phase space. The following commands should result in a figure similar to Figure 6.
Open the plot with '(*matcont*):**Window|Plot|3D-Plot**'. Choose OX=z, OY=F, OZ=y in the draw3_attributes window. Change the plotting the region to $z, y \in [-2, 2], F \in [0, 15]$.

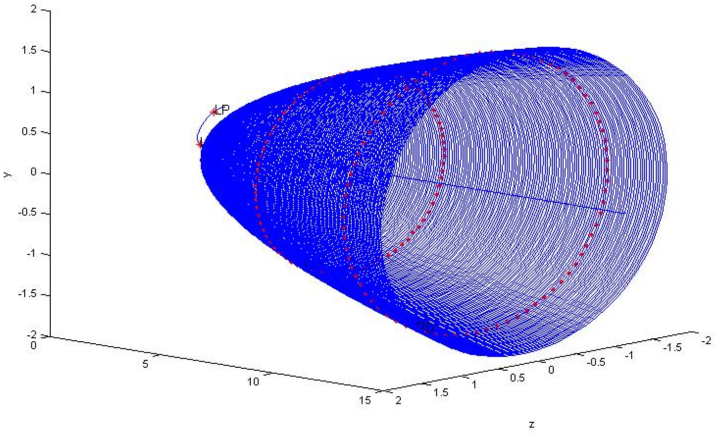


Figure 6: Equilibria and Limit cycles in (F, y, z)-space. The points and limit cycles which undergo a bifurcation are indicated with red dots.

3.2 Continuation of Cycle Bifurcations in 2 Parameters

3.2.1 Starting from a detected codim 1 bifurcation

From a PD point, we should be able to find the doubled period cycle, and also the original cycle still exists. This is just the continuation of a limit cycle in one parameter as we have done above. You may however see below that if your Point type is PD, then there are two options for the Curve type. The PD_LC would start the continuation of the double-period cycle. For the original cycle you could just use LC as Point Type. We will compute the PD-bifurcation curve in the parameters F and G. Note that the period is automatically selected as free parameter.

Execute:

- Select the PD point using '(*matcont*):**Select|Initial point**' as new initial point.
- Set the curve type to Period-doubling and set F, G as free parameters.
- Open a Numeric window to monitor the multipliers along the whole curve.
- Press '(*matcont*):**Compute|Forward**' and '(*matcont*):**Compute|Backward**'.

Results: You can now open a 2D-plot with F, G on the axes. Make sure that the plotting region for F is [0,15]. Select **Layout|Redraw diagram**. You will now see the LP, H and PD curve. The result will start to resemble Figure 8. On one part of the PD curve you have encountered a R2 bifurcation, i.e. a double multiplier -1. From this point, there is always a Neimark-Sacker bifurcation. That is missing and it is part of the next item.

3.3 Starting from a codim 2 point

Now we have also the option to start a special bifurcation curve from the Zero-Hopf point straightaway. The ZH-point has codimension two and a codimension 1 Neimark-Sacker emanates here for this case.

Execute:

- Select the ZH point as initial point and **Curve|Neimark-Sacker**.
- Now check that F and G are the two free parameters, and select to detect R4,R3,R2 along the curve.
- In the Continuer window set MaxStepsize to 0.3 and Press **Compute|Forward**.

Results: If you have the 2D-plot of the previous item still open, you will see that there is now a Neimark-Sacker bifurcation curve starting from the ZH-point all the way to the R2-point.

Note that this procedure saves a lot of time! We start directly from the codim 2 point, without any intermediate steps with Limit Cycles. It is known not to work if the bifurcating periodic orbit changes direction quickly or when the normal form coefficients are very small or very large. Then one has to compute the bifurcation point with higher accuracy, use symbolic derivatives and play with the amplitude see [6].

3.4 Visualizing your output

So far, we have already plotted some of the continuation results. It is also possible to load the output and plot and manipulate it yourself. We will give a nontrivial example of plotting the profile. This is nontrivial as the data is stored in a condensed way. So we will discuss what data we have and then come to the command line commands.

[6] Yu.A. Kuznetsov, H.G.E. Meijer, W. Govaerts and B. Sautois, Switching to nonhyperbolic cycles from codim 2 bifurcations of equilibria in ODEs, Physica D 237 No. 23 (2008) 3061-3068

Suppose we know how many mesh-points were used. This can be determined from the f variable, simply look up the first entry equal to 1. Now we need

- the index of the periodic orbit we want to plot: say ii.
- the profile of one variable, say the jj-th: x(jj:ndim:end-2,ii)
- the time-mesh tau=f(1:ntst+1,ii), with $0 = \tau_0 < \tau_1 < ... < \tau_{ntst} = 1$.
- the period T= x(end-1,ii) to scale the mesh back from [0,1] to [0,T]

So we could try to plot:

```
ntst=20;ii=4;jj=1;ndim=3;
plot(f(1:ntst+1,ii)*x(end-1,ii),x(jj:ndim:end-2,ii))
```

You will get an error message because the computed curve x contains also data about the orbit at the fine mesh-points, while you are trying to plot the profile only at the coarse mesh-points. A first correction would be to plot only the coarse grid:

```
ntst=20;ii=250;jj=1;ndim=3;ncol=4;
plot(f(1:ntst+1,ii)*x(end-1,ii),x(jj:ndim*ncol:end-2,ii))
```

This is usually not good enough, so we actually want the detailed grid. For this we need to know that between to mesh-points the orbit is stored at equi-distant time-points $t_{i,j} = t_i + \frac{j}{m}(\tau_{i+1} - \tau), \quad j = 0, 1, ..., m$. The following are the *correct* commands.

Execute:

```
load Systems\Lorenz84\diagram\H_LC(1).mat
ii=240;jj=1;ndim=3;ntst=20;ncol=4;
mesh=sort([reshape(repmat(f(1:ntst,ii),1,ncol)+...
     repmat((0:ncol-1)/ncol,ntst,1).*repmat(diff(f(1:ntst+1,ii)),1,ncol),ntst*ncol,1); 1]);
plot(mesh*x(end-1,ii),x(jj:ndim:end-2,ii))
```

Results: You

4 Connecting Orbits

In the following instructions we will see how to start the computation of homoclinic orbits. The main difficulty is to get good starting data for the continuation. We will show two methods that may be useful to accomplish this.

4.1 Homoclinic orbits starting from a limit cycle

Sometimes when following a limit cycle you will notice that the parameters do not change anymore. Meanwhile the period goes to infinity. Typically, in a graphic window plotting the continuation parameter versus the period you see something as in Figure 7right. In such cases you may suspect that you are close to a connecting orbit. You will have to look at the orbit to be sure. But suppose you notice that the orbit is close to a steady state at some point. Then it would be nice to take this as starting data for the computation of a connecting orbit. This method is implemented in Matcont, for homoclinic orbits only. The algorithm checkes when the orbit first gets closest to the initial point. This assumes that we start from a limit cycle. Along a limit cycle we can make a guess for an equilibrium by checking at which point of the cycle the vectorfield has smallest norm. This goes well for a single equilibrium, but not for two. To have a good starting limit cycle we first do a simulation. The limit cycle we get from starting at the Hopf bifurcation does not get close to a homoclinic bifurcation. So we have to find another.

Execute (Time Simulation)

- Set **Type|Initial Point|Point** and in the Starter window set $x = .1, y = 1, z = .3$ and $F = 3, G = .8$.
- In the Integrator window set Interval to 50. Open a **Window|2D graphic** with x between 0 and 2 and y between -1.5 and +1.5.
- If you **Compute|Forward** you will get something like Figure 7left.
- Now via the main window **Select|Initial Point** the last point of the orbit you just computed.
- Change the interval in the Integrator window to 10 (roughly one period) and **Compute|Forward**.
- *Intermediate Result 1* If you redraw the curve in the 2Dplot (In the 2Dplot window there is an option **Plot|Clear/Redraw curve**), it looks like we have an interesting limit cycle.

Execute (Starting Limit Cycle Continuation from a Simulation):

- In the Starter window press **Select Cycle** and change the number of test intervals to 40. Select the Period and G as free parameters (tick the boxes) and in the Continuer window set MaxStepsize to 1.0. Open a numeric window and check that the parameters and Period are displayed.
- Press **Compute|Backward** and during the continuation observe that G first goes up and settles to $G_{final} = 1.018893$. If not, select **Compute|Forward**. You can zoom in (where?!) to see that the limit cycle might be close to a saddle-focus.
- *Intermediate Result 3* At the end of the continuation, change the variables in the 2Dplot via **Layout|Variables on axes** to G and Period for the abscissa and ordinate, respectively, and change **Layout|Plotting region** the range to $[0.5, 1.5] \times [0, 100]$ to obtain Figure 7.

Execute (Starting Limit Cycle Continuation from a Simulation):

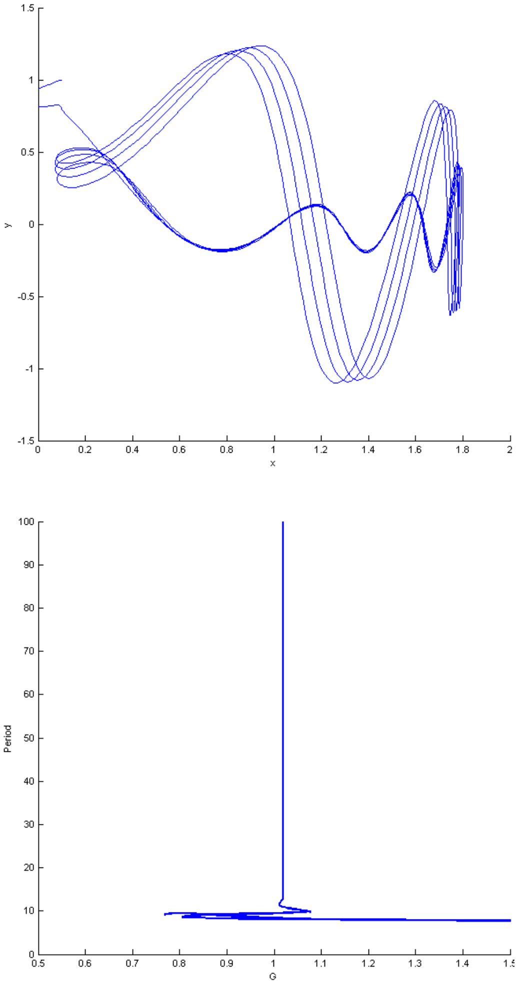


Figure 7: Left: a suitable initial orbit for finding nearby homoclinic orbits. Right: The period blows up while the parameter G does not change anymore...

- Select the last point of the LC_LC curve just computed as initial point. Next select **Curve|Homoclinic to saddle** and select F, G and eps1 as free parameters.
- In the 2Dplot change the axes to F and G with ranges [0,5] and [0,2.5], respectively. Set Maxstepsize to 3 in the continuer window.
- Press **Compute|Forward** and extend the computation at least once.

Results: You will obtain a small piece of a homoclinic bifurcation curve shown in Figure 8.

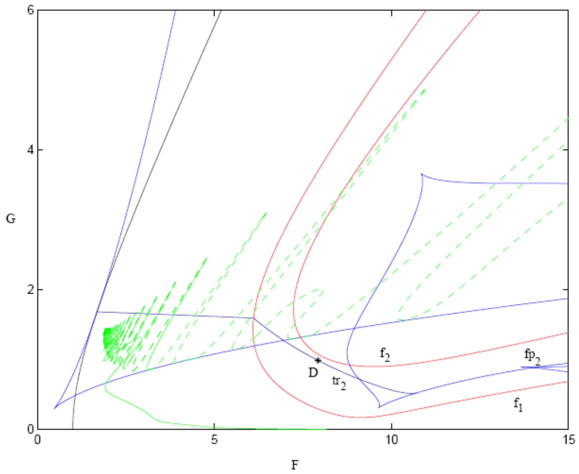


Figure 8: Bifurcation diagram taken from http://arxiv.org/abs/nlin/0111022

We see that the homoclinic bifurcation wiggles towards the ZH-point. Here two homoclinic curves come together. As a final remark we mention that you could start the continuation of the other homoclinic bifurcation by the same procedure but that at the beginning step in the limit cycle continuation you follow the branch for decreasing G. It turns out that eps1 is the correct auxilary parameter to free as the stable eigenvalues are complex. This is the hardest part for the continuation. In general, you may need to try all three of T,eps0,eps1 sequentially.

4.1.1 A useful trick

Sometimes you cannot start homoclinic continuation from a limit cycle, but still you have this limit cycle with very large period at parameters close enough to the homoclinic bifurcation. So you could also do a continuation of the limit cycle keeping the period fixed and continue in two system parameters. You will get a good indication where the homoclinic bifurcation is in parameter space.
Warning: there can be a large discrepancy between this continuation result and the actual homoclinic bifurcation.
The trick also helps to find additional starting data for the homoclinic from the other equilibrium as follows. Follow the periodic orbit with fixed period in two system parameters towards the ZH point. It will oscillate just as the true homoclinic curve, but then deviate and exhibit a cusp and then oscillate away from the ZH point. Now the final point is close to the other homoclinic.

4.2 Connecting orbits by homotopy

Execute:

- Set Initial Point|Equilibrium and select Curve|ConnectionSaddle. A new starter window with more fields below the parameters and the Integrator windows appears.
- In the Starter Window, enter $x = 0.394282, y = 0.424435, z = 1.105117$ for the saddle at $F = 6, G = 2$. Set UParam=1, eps0=.001.
- In the Integrator Window set Interval to 5.
- Open a 2D-plot with x and y on the axis. Press Compute|Forward. You will see an orbit that departs from the saddle along the unstable manifold. Then it makes a few turns, and seems to return.
- Press 'Select Connection'. A window appears with specifications for the discretization. These settings are fine, and press "OK". A new Starter window appears with parameters UParam1, SParam1. SParam1 indicates how far away the trajectory is from lying within the stable manifold. We also see T (half the integration time), and eps1 which is the distance to the saddle.
- Set G as an active parameter, and also SParam1 and eps1. Also open a numeric window, if you have not one open yet. Press Compute|Forward. In the 2D-plot you will see the continuation of the orbit piece. In the numeric window you will see that SParam1 goes to zero.
- Select the special HTHOM point as Initial point (*matcont*:Select|Initial point. Now set $G, T, eps1$ as active parameters and set MaxStepsize to 1. Compute|Forward.
- This will take a while. You will see in the 2D-plot that the orbit piece is changing a lot during continuation, while in the meantime the parameters $F, T, eps1$ are changing. But especially T is increasing as we should get closer to the saddle and this requires a longer time for the orbit piece.
- Finally you will obtain the message that eps1 is small enough. Stop the continuation. Select this last point as initial point, and set the Curve Type to "Homoclinic Saddle". Now you can do continuation in parameters F and G, with eps1 as additional free parameter as before.

Another good tutorial for the initialization by homotopy for connecting orbits, LAB5, is written by Yuri Kuznetsov. See his website or at sourceforge see lab5.pdf in files–matcont–matcont3p3. This is for travelling waves of the Fitzhugh-Nagumo equation, and very

4.3 Challenge problem

Study the bifurcations of Limit Cycles in the prey-predator system in §2.4. Also add homoclinic bifurcation curves to your diagram.

5 Bifurcations in a periodically forced system

To make this really motivating we look at a periodically forced system, the SEIR model from epidemiology. The parameter β varies periodically, but we make it autonomous as follows:

$$\left\{\begin{array}{rcl} S' & = & \mu - \mu S - \beta S I \\ E' & = & \beta S I - (\mu + \alpha) E \\ I' & = & \alpha E - (\mu + \gamma) I \\ x' & = & -2\pi y + x(1 - x^2 - y^2) \\ y' & = & 2\pi x + y(1 - x^2 - y^2) \end{array}\right. \qquad (2)$$

with $\beta = \beta_0(1 + \delta\cos(2\pi t)) = \beta_0(1 + \delta x)$ and $R = 1 - S - E - I$. Note that if you start with $x, y \neq 0$, then there it is not possible to find a Hopf point from which you can start the continuation of a Limit Cycle.

In the main matcont window: **Select|System|New** and enter all fields for the SEIR model. Note that you will be working with an equivalent system ($s = \log S$) for numerical stability. Select symbolic derivatives of order 1 and 2. Press **ok**.

We will start by integrating to observe periodic dynamics. Press *matcont*:**Type|Point** and a *starter* and *integrator* window will appear. We specify initial values in the *starter* window: ss=-3,ee=-3,ii=-3,x=1,y=0 alpha=35.482,beta=5000,gamma=100,delta=.2,mu=.02,omega=6. The system is still sensitive so you may have to decrease **MaxStepSize** to .01 in the *integrator* window. Also increase the **Interval** (integration time) to 20. Open a graphic window with *matcont*:**Window|Graphic|2Dplot**. Select the variables *ss* and *ee* on the axes. Adjust the **Layout|Plotting region** to -10 for both axes. Press *matcont*:**Compute|Forward**. You will see that after some initial excursions it will settle to a periodic orbit.

You could inspect the x, y-dynamics by changing abscissa and ordinate via **Layout|variables on axes** to see a circle.

At the bottom of the *starter* window we see a tempting button **Select Cycle**. We cannot yet push it. What the algorithm does is to find the point of the trajectory nearest to the initial value. If it finds one, it takes that the time between these points as the period and fits a mesh to the computed solution. We proceed as follows: *matcont*:**Select|Initial point** and select the last point on the curve we have just computed. In *integrator* window change the interval to 1 and press *matcont*:**Compute|Forward**. Select **Plot|Clear** and **Plot|Redraw curve** to see that it is a periodic orbit.

Now press the **Select Cycle** button in the *starter* window. A popup will appear with two fields: "Tolerance" equal to 1e-2 and "test intervals" equal to 20. Change test intervals to 40 and press **OK**. Tick the boxes for beta0 and omega as free parameters in the new *starter* window and select "yes" for Period-doubling under the heading monitor singularities. Change **MaxStepsize** to 5 in the *Continuer* window. Open a **Window**|**Numeric** window and close the figure. Press *matcont*:**Compute**|**Forward**.

If everything went well, you found two period-doubling bifurcations. Note as well in the Numeric window that omega is now really equal to 2π. Select one of them as initial point. Change **Type**|**Curve** to Perioddoubling and select beta0 and delta as free parameters.