UNIVERSITY OF TWENTE.

Matcont Tutorial

A *numerical* approach to bifurcation anaysis

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Overview

Software

Part 1: Equilibria

- Short review of bifurcations of equilibria
- Numerical Continuation

Part 2: Periodic and Connecting Orbits

- Bifurcations of Periodic orbits
- Visualization
- Connecting Orbits

Motivation

Consider a system of smooth nonlinear ODE's

$$f : \mathbb{R}^{n+m} \to \mathbb{R}^n, \quad \frac{dx}{dt} = f(x, \alpha). \tag{1}$$

- What are the equilibria? Are they stable?
- Are there any periodic orbits? Are they stable?

Not restricted to one value of α but a range of parameters: A bifurcation diagram classifies regions in parameter space with qualitatively similar dynamics.
A numerical toolbox might be very useful because f is nonlinear.

Capabilities of Auto, Content, Matcont

	A	C	M
time-integration		+	+
continuation of equilibria	+	+	+
detection of branch points and codim 1 bifurcations of equilibria	+	+	+
computation of normal forms for codim 2 bifurcations of equilibria		+	+
continuation of codim 2 equilibrium bifurcations in three parameters		+	
branch-switching from codim 2 equilibria to codim 1 bifurcations of cycles			+

Capabilities of Auto, Content, Matcont

	A	C	M
continuation of limit cycles	+	+	+
computation of phase response curve& derivative			+
detection of branch points and codim 1 bifurcations of cycles	+	+	+
continuation of codim 1 bifurcations of cycles	+		+
computation of normal forms for codim 1 bifurcations of cycles			+
detection of codim 2 bifurcations of cycles			+
computation of connecting orbits	+		+

Not better or faster than AUTO, but Matcont has a GUI and other features

General Overview of Tutorial

AIM: KNOW that such software exists and FEEL CONFIDENT that you can use it.

Skills come through experience: try, fail and learn.

Part 1 ODEs: Simulations, Numerical Continuation, Equilibria and codimension 1 bifurcations

Part 2 ODEs: Periodic orbits (cycles) and their codim 1 bifurcations, Homoclinic orbits

Part 3 Maps: Fixed points and cycles, codim 1 bifurcations

Short presentations (30 min) + 1hr Exercise
Tuesday morning part 4 is meant for questions
Also if it is about your own model/research.

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Equilibria

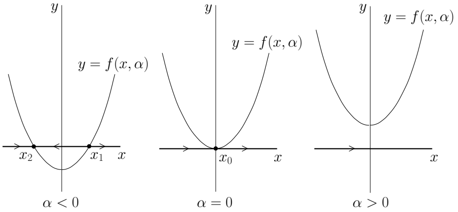
An *equilibrium* x_0 satisfies $f(x_0, \alpha) = 0$.
It is *asymptotically stable* if all the eigenvalues of $A := Df_x(x_0, \alpha)$ have negative real part.
Eigenvalues depend continuously on parameter α. Varying α, an equilibrium loses stability in two ways generically:

(a) saddle-node

(b) Hopf

Saddle-Node bifurcation

Two equilibria, one stable and one unstable, collide and disappear.



Saddle-Node bifurcation

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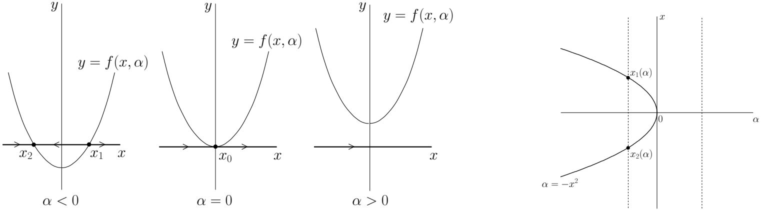
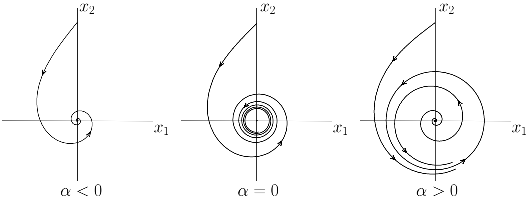


Other names: Limit Point (LP), Fold, Tangent bifurcation

Hopf bifurcation

A complex pair of eigenvalues passes through imaginary axis.
Normal form: $z' = (\alpha + i\omega)z + (c + di)z|z|^2, \quad z \in \mathbb{C}$
c is the Lyapunov coefficient.



Case $c < 0$: Supercritical Hopf, soft bifurcation
Appearance of a stable periodic orbit

Hopf bifurcation

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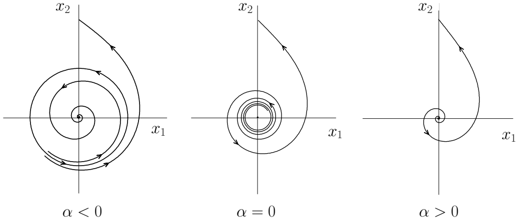

Case $c > 0$: Subcritical Hopf, hard bifurcation
Disappearance of an unstable periodic orbit

Hopf bifurcation

A complex pair of eigenvalues passes through imaginary axis.
Normal form: $z' = (\alpha + i\omega)z + (c + di)z|z|^2, \quad z \in \mathbb{C}$
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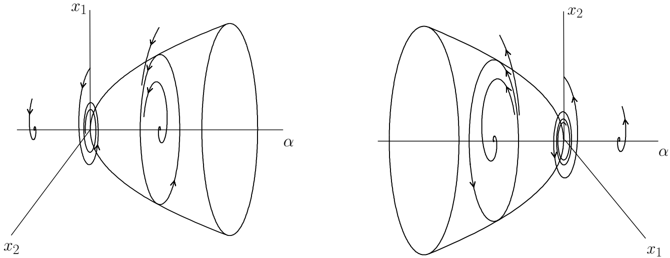


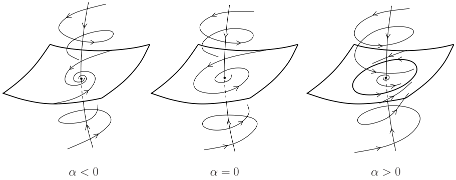
in (x, α)-space

Higher dimensions

Decompose phase space W near equilibrium into invariant unstable, center and stable manifolds:

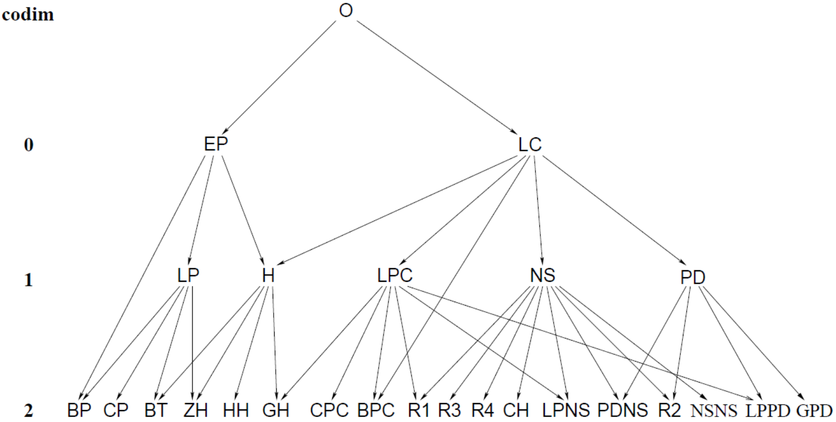
$$W = W_u \oplus W_c \oplus W_s$$

Bifurcations occur on the center manifold W_c.



In general, only look at the least stable eigenvalues.
Bifurcations still occur if W_u is non-empty.

Hierarchy of Bifurcations of Equilibria and Cycles (Labels as in MatCont)



Normal Forms

- For a Limit Point bifurcation the dynamics restricted to a 1D center manifold is given by

$$\xi' = \alpha + a\xi^2 + ..., \quad \xi \in \mathbb{R}$$

- For a Hopf bifurcation the dynamics restricted to a 2D center manifold is given by

$$z' = (\alpha + i\omega) + (c + di)z|z|^2 + ..., \quad z \in \mathbb{C}$$

When LP or H is detected, Matcont reports *a* and *c* on the Matlab command line.
Formulas for *a*, *c* are based on center-manifold reduction (not discussed here).

Numerical Continuation

Defining system F with n equations and $n+1$ variables:

$$F : \mathbb{R}^{n+1} \to \mathbb{R}^n, \quad F(x, \alpha) = 0.$$

We assume $rank(Df_{x,\alpha}) = n$, i.e. a regular system.
By the Implicit Function Theorem this defines a curve.

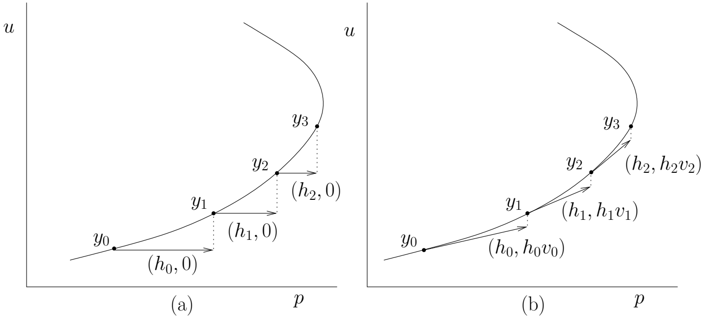
Example: hyperbolic equilibria $f(x, p) = 0$.
Locally, we find a curve $x(\alpha)$, since $rank(Df_x) = n$.

For numerical approximations of the curve:

- Fix a component, e.g. the parameter
- Use additional equation, pseudo-arclength condition

Numerical Continuation Algorithms

Fixing the parameter at every step
Without or with tangent vector



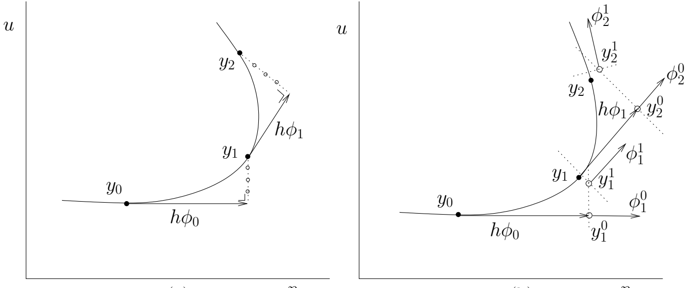
Initial point y_0 → Predict new point $\tilde{y}_1$ → Newton corrections to obtain y_1

Numerical Continuation Algorithms

Search for new point in space orthogonal to tangent vector

$$\langle \phi_0, \tilde{y}_1 - (y_0 + h\phi_0) \rangle = 0.$$

Pseudo-Arclength Moore-Penrose



(a) (b)

Matcont uses Moore-Penrose, but you could switch.

Initial point $y_0 \rightarrow$ Predict new point $\tilde{y}_1 \rightarrow$ Newton corrections to obtain y_1

Continuation of equilibria in 1 parameter

We need

- a system $x' = f(x, \alpha)$.
- an initial point $y_0 = (x_0, \alpha_0)$ such that $f(x_0, \alpha_0) \approx 0$.
- a continuation program.
- assign one parameter to be free, i.e. allow it to vary.
- monitor test functions $h(x, p)$ to detect bifurcations.

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- a system $x' = f(x, \alpha)$.
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- a continuation program.
- assign one parameter to be free, i.e. allow it to vary.
- monitor test functions $h(x, p)$ to detect bifurcations.

Test functions; not based on eigenvalues directly

- Limit Point: $h(x, \alpha) = \phi(end)$. This uses the IFT!
- Hopf: $h(x, \alpha) = 2A \odot I$.
  If $A = Df_x(x_0)$ has eigenvalues $\lambda_{1\ldots n}$, then the bi-alternate product $2A \odot I$ has eigenvalues $\lambda_i + \lambda_j, \ \ 1 \leq i < j \leq n$.

Continuation of bifurcations in 2 parameters

Add more conditions and auxilary variables to the defining system

$$F : \mathbb{R}^{n+\tilde{n}+2} \to \mathbb{R}^{n+\tilde{n}+1}, \quad F = \begin{pmatrix} f(x,\alpha) \\ s(x,\alpha) \end{pmatrix} = 0.$$

$s(x,p)$ is a function defining a Limit Point or Hopf bifurcation.

For a Limit Point $A = Df$ has rank deficiency 1. Define s as the solution of a bordered system

$$\begin{pmatrix} A & p \\ q^T & 0 \end{pmatrix} \begin{pmatrix} w(x,\alpha) \\ s(x,\alpha) \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

with bordering vectors that approximate the true nullspace
$Aq_0 = A^T p_0 = 0$ and $\|q\| = \langle p, q \rangle = 1$
At a fold bifurcation $s(x_0, \alpha_0) = 0$.

Continuation of bifurcations in 2 parameters

Add more conditions and auxilary variables to the defining system

$$F : \mathbb{R}^{n+\tilde{n}+2} \to \mathbb{R}^{n+\tilde{n}+1}, \quad F = \begin{pmatrix} f(x,\alpha) \\ s(x,\alpha) \end{pmatrix} = 0.$$

$s(x,p)$ is a function defining a Limit Point or Hopf bifurcation.

For a Hopf bifurcation $A^2 + \omega^2 I$ has rank deficiency 2. Define s as two independent components of g obtained from

$$\begin{pmatrix} A^2 + \kappa I & p_1 & p_2 \\ q_1^T & 0 & 0 \\ q_2^T & 0 & 0 \end{pmatrix} \begin{pmatrix} w(x,\alpha) \\ g(x,\alpha) \end{pmatrix} = \begin{pmatrix} 0_{n\times 2} \\ I_2 \end{pmatrix},$$

with auxilary variable $\kappa = \omega^2$ and bordering vectors not orthogonal to $Null(A^2 + \omega^2 I)^{T(*)}$.
At a Hopf bifurcation $g_{ij}(x_0, \alpha_0) = 0, \quad i,j = 1,2.$

Codim 2 points are organizing centers

Codim 2 bifurcation if normal form coefficient vanishes or additional critical eigenvalue.
Locus of new bifurcation curves.

- Cusp; normal form coefficient $a = 0$.
- Bogdanov-Takens (BT); double zero eigenvalue.
- Degenerate Hopf (GH); Lyapunov coefficient $c = 0$.
- Zero-Hopf; eigenvalue 0 and imaginary pair $\pm i\omega$.
- Double Hopf; two imaginary pairs of eigenvalues

Tutorial: Part 1

Some general remarks:

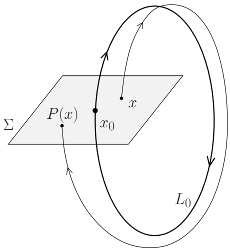
- Never forget to do simulations as well.
- The continuation adapts stepsize; smaller steps near folds.
- Setting stepsizes for the continuation or initializers requires experience.

Tutorial §2: Defining a system and Simulations
Continuation of Equilibria and
codim 1 bifurcations of Equilibria

Overview

Periodic Orbit $\sim$ Limit Cycle

A *Periodic Orbit* satisfies $x(t+T) = x(t)$ for a minimal *period* $T > 0$. The stability of the cycle is given by its *Floquet multipliers* μ:
There is always a trivial multiplier $\mu_1 = 1$. The cycle is stable if $|\mu_i| < 1, \quad i = 2, ..., n$. Typically determined as the eigenvalues of the linearization of the Poincaré map.



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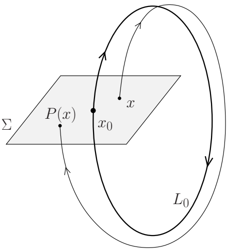
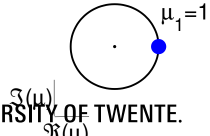


The cycle may loose stability as upon changing a parameter a multiplier crosses the unit circle: Limit Point bifurcation



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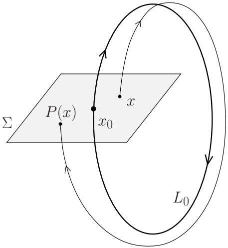
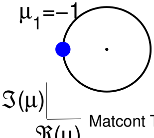
There is always a trivial multiplier $\mu_1=1$.
The cycle is stable if $|\mu_i|<1, \quad i=2,\ldots,n$. Typically determined as the eigenvalues of the linearization of the Poincaré map.



The cycle may loose stability as upon changing a parameter a multiplier crosses the unit circle: Period-Doubling bifurcation



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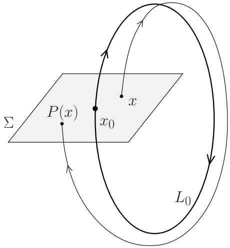
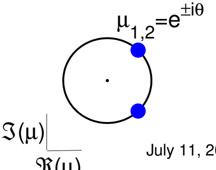
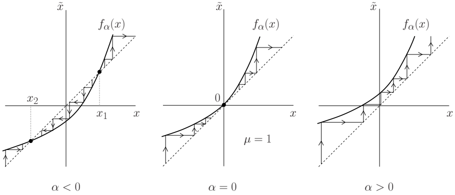


The cycle may loose stability as upon changing a parameter a multiplier crosses the unit circle: Neimark-Sacker bifurcation



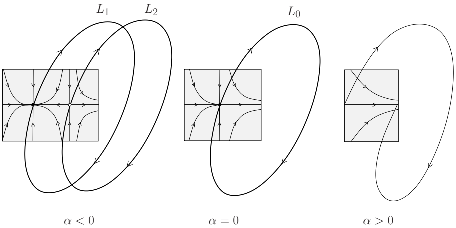
Limit Point of Cycles (LPC)



$$\xi \mapsto \alpha + \xi + a\xi^2$$

Two periodic orbits collide and disappear.

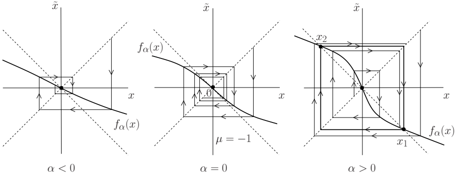
Limit Point of Cycles (LPC)



$$\xi \mapsto \alpha + \xi + a\xi^2$$

Two periodic orbits collide and disappear.

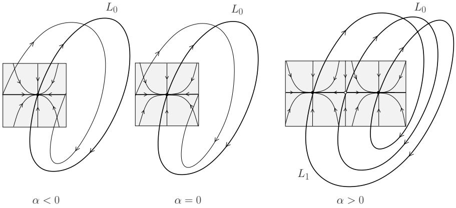
Period-doubling (PD)



$$\xi \mapsto (-1 + \alpha)\xi + b\xi^3$$

The cycle becomes unstable and a cycle of double period is born.

Period-doubling (PD)



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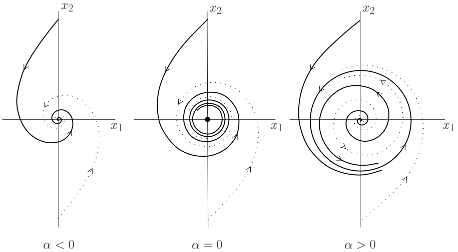
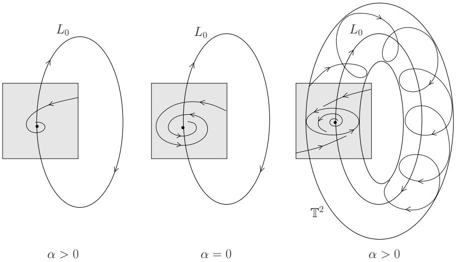
Neimark-Sacker (NS)



$$z \mapsto e^{i\theta(\alpha)}\left((1+\alpha)z + (c+di)z|z|^2\right)$$

The cycle becomes unstable and a torus appears around the cycle.

Neimark-Sacker (NS)



$$z \mapsto e^{i\theta(\alpha)} \left((1+\alpha)z + (c+di)z|z|^2\right)$$

The cycle becomes unstable and a torus appears around the cycle.

Limit Cycles: Defining systems

Periodic orbits $x(t) = x(t + T)$ are computed with a Boundary Value Problem:

- Time rescaling $T = 1$ and divide $t \in [0, 1]$ into N little intervals: $0 < t_1 < ... < t_N = 1$.
- On each interval approximate solution x by polynomial p_i.
- Polynomial should satisfy the ODE at (Gaussian) collocation points.
- Glue the little intervals $p_i(t = 1) = p_{i+1}(t = -1)$.
- Periodicity requires $x(0) = p_1(-1) = p_N(1) = x(1)$.
- Phase condition for a unique solution.
- Continuation variables x_i, 1 parameter, period T.

Limit Cycles: Collocation

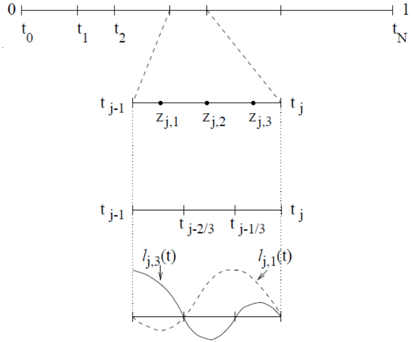


Figure 23: The mesh $\{0 = t_0 < t_1 < \cdots < t_N \;=\; 1\}$. Collocation points and "extended-mesh points" are shown for the case $m = 3$, in the jth mesh interval. Also shown are two of the four local Lagrange basis polynomials.

Limit Cycle Continuation

Initial data for continuation:

- From a Hopf bifurcation there is a one-parameter family of periodic orbits. Use linear center-manifold approximation to start Limit Cycle continuation from a Hopf bifurcation: $x = x_0 + \varepsilon \Re(e^{i\omega_0 t} q_0), \ \alpha = \alpha_0$.
- Start LC continuation from simulated (periodic) orbit (if there is no Hopf nearby)

When LC continuation fails, e.g.:
c is very small or large, close to a saddle-node, stiff system
Solutions:

- ”Play” with the amplitude ε.
- Use more mesh points.

More on Limit Cycles

- Detection of LP, PD and NS points; test-functions use linearization.
  Switched off by default for speed, and spurious detections.
- Computation of the normal form coefficients a_{LP}, b_{PD}, c_{PD}; reported on the Matlab command line.
- Continuation of LP, PD and NS in 2 parameters; additional equations defined by bordered systems.
- Detection of codim 2 bifurcations of cycles;
  Defined by additional critical multipliers or degenerate normal form coefficients.
  Normal form coefficients are computed.

Checking output

Understand your model and check your results

- 2D/3D graphic:
  Variables (all/max/min), parameters, period
  (close during continuation for speed)
- Numeric window:
  Variables, parameters, period, stepsize, testfunctions

Loading output

All data is stored in a folder “diagram”. This allows inspection afterwards. For each curve we store:

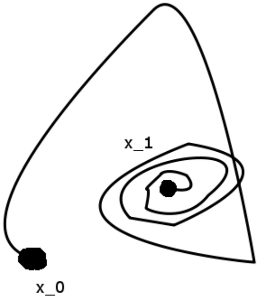
- *x*: the variables, phase space coordinates, system parameters and auxilary variables.
- *v*: The tangent vector to the curve.
- *s*: structure with info about special points: First/last and type of bifurcations.
- *h*: # Newton corrections, Stepsize, values of testfunctions.
- *f*: (for LC: the MESH), Eigenvalues/Multipliers.

Definition of connecting orbits

Take two saddle steady states x_0 and x_1 and an orbit $x(t)$.
$x(t)$ is a connecting orbit if

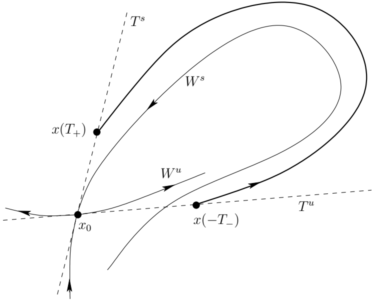
$$\lim_{t \to -\infty} x(t) = x_0 \quad and \quad \lim_{t \to +\infty} x(t) = x_1$$

If $x_0 = x_1$ then homoclinic, if $x_0 \neq x_1$ then heteroclinic.



Eigenspaces

Another way to look at it:
$\lim_{t\to-\infty} x(t) = x_0$ means $x(-T) \in W^u(x_0)$
$\lim_{t\to+\infty} x(t) = x_1$ means $x(+T) \in W^s(x_1)$
or rather orthogonal to the complement!



We cannot compute infinite trajectories...

Defining system for connecting orbits

$$
\begin{cases}
\dot{x}(t) - f(x(t), p) = 0, & \text{orbit piece} \\
f(x_0) = 0, & \text{equilibrium} \\
f(x_1) = 0, & \text{equilibrium} \\
\int_{-T}^{T} (x(t) - x_0(t))^T \dot{x}_0(t) dt = 0, & \text{phase condition} \\
L_s(p)(x(-T) - x_0) = 0, & \text{left boundary projection} \\
L_u(p)(x(T) - x_1) = 0, & \text{right boundary projection} \\
\|x(T) - x_0\| - \varepsilon_0 = 0, & \text{distance to } x_0 \\
\|x(T) - x_1\| - \varepsilon_1 = 0, & \text{distance to } x_1
\end{cases}
$$

Connecting orbits are a codimension 1 phenomenon:
Two free system parameters and 1(2) auxilary variable(s) from $T, \varepsilon_0, \varepsilon_1$: choice depends on the system.

How to start continuation?

It is nice that we have defining systems, but how do we give good initial data for continuation:

- Equilibrium: from “any” point
- Limit cycle: from Hopf or a simulation
- Bifurcation: At points detected during continuation
- Connecting orbits...

Methods to start homoclinic continuation

0. An analytic approximation if available: For Bogdanov-Takens only.
1. Start from limit cycle with large period.
2. Homotopy in several steps:
    1. Simulation starting in unstable manifold of a saddle x_0.
    2. Take orbit piece that came closest to target saddle x_1.
    3. Bring the endpoint of the orbit piece into the stable eigenspace of target equilibrium x_1
    4. Bring the endpoint close enough to x_1.

Homoclinic bifurcations

We have not covered bifurcations of homoclinic orbits!
Matcont supports detection of these bifurcations.
Good texts for reference:

- Chapter 6 of book by Yuri Kuznetsov
- Handbook chapter by Sandstede and Homburg: google for "Homoclinic and Heteroclinic Bifurcations in Vector Fields"

Tutorial: Part 2

- Tutorial §3: Limit Cycles in Lorenz84 and plotting
- Tutorial §4: Homoclinic orbit continuation.